ASYMPTOTIC ANALYSIS FOR SCATTERING OFF ONE-DIMENSIONAL COMPACTLY SUPPORTED WEAK POTENTIALS

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ABSTRACT. For a compactly supported $q \in L^1(0,\infty)$, consider $-u'' - k^2 qu = k^2 u$ on the half-line with $u'(0) = \nu u(0)$. The boundary quotient defined by the regular solution has a removable threshold singularity and a convergent low-energy series; an explicit Volterra recursion computes every coefficient. For real data we prove an exact finite-difference identity for two coefficients and determine the optimal Lipschitz modulus on real L^1-balls:

$$\mathfrak{L}_M(k,\nu) = \frac{2k^3}{k^2+\nu^2} \left\| \cos(k\,\cdot) + \frac{\nu}{k}\sin(k\,\cdot) \right\|_\infty^2 \left(1 + O_{M,R}(k)\right),$$

uniformly for all real ν. This is the sharp crossover between cubic fixed-Robin and linear Neumann sensitivity.

The principal result treats families outside bounded L^1-sets. If real V, W satisfy $q_k = k^{-1}V + W + o_{L^1}(1)$ and $\nu(k) = \alpha k + \beta k^2 + o(k^2)$, write $Q_j(f) = \int_0^R x^j f(x)\,\mathrm{d}x$, let $\mathcal{J}$ denote the ordered quadratic form defined below, and put

$$a = \alpha - Q_0(V), \qquad \Gamma = Q_0(W) - \beta + \big(2\alpha - Q_0(V)\big)Q_1(V) - \mathcal{J}(V).$$

Then

$$S_{\nu(k)}(k; q_k) = \frac{\mathrm{i} - a}{\mathrm{i} + a} + \frac{2\mathrm{i}\Gamma}{(a+\mathrm{i})^2}k + o(k).$$

For $q_k = k^{-\gamma}V$, the generic critical exponent is $\gamma = 1$. If $V \neq 0$ is real and $Q_0(V) = 0$, it shifts to $\gamma = 3/2$, where $\mathcal{J}(V) = -\int_0^R \big(\int_0^x V(t)\,\mathrm{d}t\big)^2\,\mathrm{d}x < 0$. The same normal form resums the leading term of every Born coefficient.

1. INTRODUCTION

Let $R > 0$ and let

$$\mathcal{Q}_R := \big\{ q \in L^1(0,\infty;\mathbb{C}) : q = 0 \text{ almost everywhere on } (R,\infty) \big\}.$$

For $q \in \mathcal{Q}_R$, a spectral parameter $k \in \mathbb{C}$, and a boundary parameter $\nu \in \mathbb{R}$, we consider

$$-u''(x) - k^2 q(x) u(x) = k^2 u(x), \qquad x > 0, \tag{1.1}$$

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subject to

$$u'(0) = \nu u(0). \tag{1.2}$$

Equation (1.1) is a one-dimensional Sturm–Liouville equation whose compact perturbation depends quadratically on the spectral parameter. It is equivalent to

$$u'' + k^2(1+q)u = 0.$$

The weak-potential regime in the title means that

$$q = \varepsilon V, \qquad V \in \mathcal{Q}_R, \tag{1.3}$$

with ε near zero. Most of the analysis is first carried out for an arbitrary $q \in \mathcal{Q}_R$; the specialization (1.3) then produces a convergent perturbation theory in the coefficient as well as in k.

For $x > R$, equation (1.1) is free, so every solution is a linear combination of e^{ikx} and e^{-ikx} when $k \neq 0$. We define $S_\nu(k;q)$ by requiring a nonzero solution of (1.1)–(1.2) to have the normalization

$$u(x,k;q) = e^{ikx} + S_\nu(k;q)e^{-ikx}, \qquad x > R. \tag{1.4}$$

For real $k \neq 0$, real q, and real ν, this normalization is always possible and $|S_\nu(k;q)| = 1$. We use the term *scattering coefficient* for this scalar quotient, but the arguments below are formulated entirely in terms of regular solutions, Wronskians, Volterra operators, and holomorphic maps between Banach spaces. For complex-valued q, this normalization is used only when the coefficient of e^{ikx} is nonzero; elsewhere S_ν is understood through the meromorphic boundary quotient derived below.

The low-energy point $k = 0$ is singular at the level of the exterior basis, because the two exponentials coalesce. It is nevertheless regular for the quotient after one identifies the correct common factor. The distinction between $\nu \neq 0$ and $\nu = 0$ is encoded in the zero-energy regular solution $1 + \nu x$. The main results of the paper can be summarized as follows.

(1) The scattering coefficient has the exact representation

$$S_\nu(k;q) = e^{2ikR}\frac{ikF_\nu(k;q) - G_\nu(k;q)}{ikF_\nu(k;q) + G_\nu(k;q)},$$

where F_ν and G_ν are the value and derivative at R of the regular solution.

(2) The maps F_ν and G_ν are even entire functions of k. For $q = \varepsilon V$, they are jointly entire in (k, ε). The quotient is holomorphic near $k = 0$ after cancellation of a single factor in the Neumann case. Its expansion is therefore convergent and admits ordinary Cauchy remainder estimates.

(3) A Volterra recursion computes the regular-solution coefficients, and a quotient recursion then computes every Taylor coefficient of S_ν. The first terms are given explicitly through cubic order. In the Neumann case the cubic coefficient contains a quadratic ordered moment of q.

(4) The map $q \mapsto S_\nu(k;q)$ is Fréchet holomorphic on every region on which the denominator is nonzero. Its derivative in the direction $h \in \mathcal{Q}_R$ is

$$D_q S_\nu(k;q)[h] = \frac{2\mathrm{i}k^3 e^{2\mathrm{i}kR}}{(\mathrm{i}kF_\nu + G_\nu)^2} \int_0^R h(x)\phi_\nu(x,k;q)^2 \,\mathrm{d}x.$$

We also prove the exact nonlinear identity

$$S_\nu(k;q_1) - S_\nu(k;q_0) = \frac{2\mathrm{i}k^3 e^{2\mathrm{i}kR}}{D_1 D_0} \int_0^R (q_1 - q_0)\phi_1\phi_0 \,\mathrm{d}x,$$

and its simultaneous boundary-parameter extension. These formulas give the first Born term, phase monotonicity, and exact coefficient–boundary compensation.

(5) On the real L^1-ball of radius M, the optimal global Lipschitz modulus is the supremum of the exact derivative norm and satisfies, uniformly for every real ν,

$$\mathfrak{L}_M(k,\nu) = \frac{2k^3}{k^2+\nu^2} \left\| \cos(k\,\cdot) + \frac{\nu}{k}\sin(k\,\cdot) \right\|_\infty^2 \big(1 + O_{M,R}(k)\big).$$

We determine its next coefficient when $\nu = \alpha k + \beta k^2$.

(6) If $\nu/k \to \alpha \in \mathbb{R}$ while q remains L^1-bounded, then

$$S_\nu(k;q) \longrightarrow \frac{\mathrm{i}-\alpha}{\mathrm{i}+\alpha}.$$

For the exact scale $\nu = \alpha k$, the next term is

$$\frac{2\mathrm{i}Q_0}{(\alpha+\mathrm{i})^2}k, \qquad Q_0 = \int_0^R q.$$

Thus the fixed-Robin and Neumann expansions are two nonuniform limits of one two-parameter law.

(7) For real V, W in the singular family $q_k = k^{-1}V + W + o_{L^1}(1)$ and $\nu(k) = \alpha k + \beta k^2 + o(k^2)$, the limiting Cayley parameter is renormalized from α to $\alpha - \int V$, and we compute the first profile-sensitive correction. For a real nonzero profile in the power family $q_k = k^{-\gamma}V$, the resulting critical exponent is 1 when $\int V \neq 0$, and $3/2$ when $\int V = 0$.

Compactly supported one-dimensional coefficient problems and their inverse maps are classical; see, for example, [5, 3, 17]. Analytic expansions of regular Sturm–Liouville solutions in the spectral parameter, including recursive spectral-parameter power series, are also well established [11, 12]. Related energy-dependent inverse problems are studied in [8, 7, 10]. For low-energy scattering in one dimension, see [4]; for half-line operators with general self-adjoint endpoint conditions, see [1, 2]. Robin endpoint functions also occur in [6]. Low-frequency expansions for one-dimensional Helmholtz equations, including half-line problems with frequency-dependent Robin data, were obtained by Loran and Mostafazadeh [13]. After translating conventions—in particular, their reflection coefficient is S_ν^{-1} in the present normalization—their equation (61) in the bounded, piecewise-continuous coefficient class contains the transition limit and the first two bounded-coefficient transition terms recorded below. We therefore do not claim novelty for those coefficients, nor for spectral-parameter power series or Fréchet linearization as such. Our Volterra formulation places the quotient calculus in an L^1 setting and supplies the exact finite perturbation identity and sharp stability constants.

The linear energy-dependent pencil underlying the singular family is part of the model class studied by Hryniv and Manko [7], under different structural hypotheses; we do not claim invention of that pencil. The singular critical families in Section 10.1, for which $k^2(1+q_k) = kV + O(k^2)$, lie outside the locally uniformly bounded coefficient hypothesis used in [13]. For this half-line quotient, the author is not aware of prior statements of the coefficient–boundary double-scaling normal form, its ordered-moment correction, the mean-zero critical class, or the sharp L^1 crossover modulus. General background may be found in [15, 18].

The paper is organized as follows. Section 2 records the weighted Sturm–Liouville interpretation and the zero-energy resonance. Section 3 derives the exact quotient. Section 4 proves analytic dependence and quantitative Volterra bounds. Section 5 resolves the threshold. Sections 6 and 7 give the all-order recursion and explicit coefficients. Section 8 develops exact coefficient variation, phase monotonicity, and sharp stability; Section 9 treats the Born expansion. Section 10 first recalls the bounded-coefficient Robin–Neumann transition and then proves the singular critical-contrast results. Section 11 relates convergence radii to zeros of the boundary denominator. An exactly solvable coefficient is considered in Section 12, and Section 13 records consequences and limitations. The appendices contain coefficient algebra and a weighted-Sobolev a posteriori estimate.

2. Weighted Sturm–Liouville realization

The scattering coefficient can be developed under the minimal assumption $q \in \mathcal{Q}_R$. Before doing so, it is useful to record the operator-theoretic meaning of the threshold when the coefficient has the additional properties

$$q \in L^\infty(0, R; \mathbb{R}), \qquad w := 1 + q \geq c_0 > 0 \quad \text{almost everywhere.} \tag{2.1}$$

In this subsection only, let

$$\mathcal{H}_w := L^2((0,\infty), w(x)\,\mathrm{d}x).$$

The norms of $\mathcal{H}_w$ and $L^2(0,\infty)$ are equivalent because w is bounded above and below and equals one for $x > R$.

For $\nu \in \mathbb{R}$, consider the form

$$\mathfrak{a}_\nu[u,v] := \int_0^\infty u'(x)\overline{v'(x)}\,\mathrm{d}x + \nu u(0)\overline{v(0)}, \qquad \mathrm{dom}(\mathfrak{a}_\nu) = H^1(0,\infty). \tag{2.2}$$

The trace inequality

$$|u(0)|^2 \leq \delta \|u'\|_{L^2}^2 + C_\delta \|u\|_{L^2}^2, \qquad \delta > 0, \tag{2.3}$$

shows that $\mathfrak{a}_\nu$ is densely defined, closed, and bounded from below in $\mathcal{H}_w$. The first representation theorem [9] therefore determines a self-adjoint operator $H_{\nu,w}$.

Proposition 2.1 (Operator realization)**.** *Under* (2.1)*, the operator associated with* (2.2) *is*

$$H_{\nu,w} u = -w^{-1} u'', \tag{2.4}$$

with domain

$$\begin{aligned} \operatorname{dom}(H_{\nu,w}) = \big\{ u \in H^1(0,\infty) : \ & u' \in AC_{\mathrm{loc}}[0,\infty), \\ & w^{-1}u'' \in \mathcal{H}_w, \quad u'(0) = \nu u(0) \big\}. \end{aligned} \tag{2.5}$$

Moreover,

$$\sigma_{\mathrm{ess}}(H_{\nu,w}) = [0,\infty),$$

and the spectrum below zero is discrete with finite multiplicities and can accumulate only at zero.

Proof. If u lies in the operator domain supplied by the representation theorem, testing against compactly supported smooth functions gives $H_{\nu,w}u = -w^{-1}u''$ distributionally. The stated local regularity follows in one dimension. Integration by parts gives

$$\int_0^\infty (-u'')\overline{v}\ \mathrm{d}x = \int_0^\infty u'\overline{v'}\ \mathrm{d}x + u'(0)\overline{v(0)}.$$

Comparison with (2.2) yields $u'(0) = \nu u(0)$, and the converse follows from the same identity.

Impose an additional Dirichlet condition at R. The resulting decoupled operator is the orthogonal sum of a regular Sturm–Liouville operator on $(0, R)$, which has compact resolvent, and the free Dirichlet Laplacian on (R,∞). Restoring the two matching conditions at R changes the resolvent by an operator of rank at most two. The essential spectrum is therefore $[0,\infty)$. Standard self-adjoint spectral theory then implies that the spectrum outside $[0,\infty)$ is discrete with finite multiplicities. Semiboundedness excludes accumulation at $-\infty$, so its only possible accumulation point is 0; see [16, 15]. □

The spectral equation $H_{\nu,w}u = k^2 u$ is precisely

$$-u'' = k^2 w u,$$

which is (1.1). At $k = 0$, every solution is affine:

$$u(x) = a + bx.$$

Imposing the Robin condition gives $b = \nu a$. This elementary observation distinguishes the Neumann endpoint.

Proposition 2.2 (Zero-energy resonance)**.** *Under* (2.1)*, the differential equation*

$$-u'' = 0$$

has a nonzero bounded solution satisfying $u'(0) = \nu u(0)$ if and only if $\nu = 0$. In that case every bounded solution satisfying the boundary condition is constant.

Proof. At zero energy the coefficient w disappears from the differential equation, so $u = a + bx$. The boundary condition gives $b = \nu a$. If $\nu = 0$, any nonzero constant is a bounded solution. If $\nu \neq 0$, a nonzero solution satisfying the boundary condition grows linearly; the only bounded solution is zero. $\square$

Remark 2.3*.* The bounded Neumann solution is not an element of $\mathcal{H}_w$ and hence is not an eigenfunction of $H_{\nu,w}$. In the one-dimensional terminology used here, it is a zero-energy resonance. The different values $S_\nu(0;q) = -1$ for $\nu \neq 0$ and $S_0(0;q) = 1$ derived below are the quotient-theoretic manifestation of this resonance. No positivity assumption such as (2.1) is needed for the analytic proofs.

3. The regular solution and the boundary quotient

We use the following minimal notion of solution. A function u on $[0,\infty)$ is a solution of (1.1) if $u \in C^1([0,\infty))$, its first derivative is locally absolutely continuous, and the equation holds almost everywhere. These conditions are natural for $q \in L^1_{\mathrm{loc}}$.

Definition 3.1 (Regular solution)**.** For $q \in \mathcal{Q}_R$ and $k, \nu \in \mathbb{C}$, let $\phi_\nu(\,\cdot\,, k; q)$ be the solution determined by

$$\phi_\nu(0,k;q) = 1, \qquad \phi_\nu'(0,k;q) = \nu. \tag{3.1}$$

Its endpoint data are denoted by

$$F_\nu(k;q) := \phi_\nu(R,k;q), \qquad G_\nu(k;q) := \phi_\nu'(R,k;q). \tag{3.2}$$

Existence and uniqueness follow either from standard linear ordinary differential equation theory or directly from the Volterra construction in Section 4. For $x > R$ and $k \neq 0$, write

$$\phi_\nu(x,k;q) = A_\nu(k;q)e^{\mathrm{i}kx} + B_\nu(k;q)e^{-\mathrm{i}kx}. \tag{3.3}$$

Proposition 3.2 (Exact quotient formula)**.** *For $k \neq 0$,*

$$A_\nu(k;q) = \frac{e^{-\mathrm{i}kR}}{2\mathrm{i}k}\big(\mathrm{i}kF_\nu(k;q) + G_\nu(k;q)\big), \tag{3.4}$$

$$B_\nu(k;q) = \frac{e^{\mathrm{i}kR}}{2\mathrm{i}k}\big(\mathrm{i}kF_\nu(k;q) - G_\nu(k;q)\big). \tag{3.5}$$

If q, ν, k are real and $k \neq 0$, then $A_\nu(k;q) \neq 0$, the normalization (1.4) *is unique, and*

$$S_\nu(k;q) = e^{2\mathrm{i}kR}\frac{\mathrm{i}kF_\nu(k;q) - G_\nu(k;q)}{\mathrm{i}kF_\nu(k;q) + G_\nu(k;q)}. \tag{3.6}$$

Proof. Matching (3.3) and its derivative at R gives

$$\begin{pmatrix} e^{\mathrm{i}kR} & e^{-\mathrm{i}kR} \\ \mathrm{i}ke^{\mathrm{i}kR} & -\mathrm{i}ke^{-\mathrm{i}kR} \end{pmatrix} \begin{pmatrix} A_\nu \\ B_\nu \end{pmatrix} = \begin{pmatrix} F_\nu \\ G_\nu \end{pmatrix}.$$

The determinant is $-2\mathrm{i}k$, and inversion yields (3.4)–(3.5). If the data are real and $\mathrm{i}kF_\nu + G_\nu = 0$, then its real and imaginary parts give $F_\nu = G_\nu = 0$, which contradicts uniqueness for the Cauchy problem at R. Thus $A_\nu \neq 0$. Dividing the regular solution by A_ν gives (1.4), with $S_\nu = B_\nu / A_\nu$, and proves (3.6). □

Proposition 3.3 (Reality symmetry and unitarity)**.** *Let $q \in \mathcal{Q}_R$ and $\nu \in \mathbb{R}$ be real. For every real $k \neq 0$,*

$$|S_\nu(k;q)| = 1, \qquad S_\nu(-k;q) = \overline{S_\nu(k;q)} = S_\nu(k;q)^{-1}. \tag{3.7}$$

Proof. The regular solution and both endpoint data are real for real parameters. The numerator in (3.6) is the negative of the complex conjugate of the denominator. The exponential prefactor has modulus one, and hence $|S_\nu| = 1$. The equation and the regular data are unchanged by replacing k with $-k$, so F_ν and G_ν are even functions of k. Substitution in (3.6) gives the remaining identities. □

Corollary 3.4 (The free endpoint)**.** *For $q = 0$,*

$$S_\nu^{\mathrm{free}}(k) := S_\nu(k;0) = \frac{\mathrm{i}k - \nu}{\mathrm{i}k + \nu}. \tag{3.8}$$

In particular, $S_0^{\mathrm{free}} \equiv 1$.

Proof. The free regular solution is

$$\phi_\nu^{\mathrm{free}}(x,k) = \cos(kx) + \frac{\nu}{k}\sin(kx), \tag{3.9}$$

where the quotient is continued at $k = 0$. A direct calculation gives

$$\mathrm{i}kF_\nu^{\mathrm{free}}(k) + G_\nu^{\mathrm{free}}(k) = (\nu + \mathrm{i}k)e^{\mathrm{i}kR}. \tag{3.10}$$

Formula (3.8) follows from (3.6). □

Remark 3.5 (Choice of support bound). The number R need not be the smallest support radius. If R is replaced by a larger number, the endpoint vector is propagated through a free interval. Formula (3.6) is unchanged because the resulting phase is exactly cancelled by $e^{2\mathrm{i}kR}$.

Remark 3.6 (Dirichlet endpoint). The Dirichlet condition $u(0) = 0$ is handled by the regular data $\phi(0,k) = 0$, $\phi'(0,k) = 1$. Formula (3.6) remains valid and gives $S_{\mathrm{D}}(0) = -1$. Dirichlet data are a separate boundary condition, not literally a finite Robin parameter, although suitable large-$|\nu|$ limits recover them.

4. Volterra operators and analytic dependence

Integrating the differential equation twice and imposing (3.1) gives

$$\phi_\nu(x,k;q) = 1 + \nu x - k^2 \int_0^x (x-t)(1+q(t))\phi_\nu(t,k;q)\,\mathrm{d}t. \tag{4.1}$$

Let $X = C([0,R])$, with the supremum norm, and define

$$(\mathcal{V}_q f)(x) := \int_0^x (x-t)(1+q(t))f(t)\,\mathrm{d}t. \tag{4.2}$$

For $x \in [0,R]$, put

$$M_q(x) := \int_0^x |1+q(t)|\,\mathrm{d}t, \qquad M_q := M_q(R).$$

Lemma 4.1 (Iterated Volterra bounds)**.** *For $m \geq 1$, $f \in X$, and $x \in [0,R]$,*

$$|(\mathcal{V}_q^m f)(x)| \leq \|f\|_\infty \frac{R^m M_q(x)^m}{m!}, \tag{4.3}$$

$$|(\mathcal{V}_q^m f)'(x)| \leq \|f\|_\infty \frac{R^{m-1} M_q(x)^m}{m!}. \tag{4.4}$$

Proof. After m iterations, the integration variables lie in the ordered simplex

$$0 < t_m < \cdots < t_1 < x.$$

Every difference introduced by a kernel $x-t$ is bounded by R. Removing these differences leaves the product measure $\prod_{j=1}^m |1+q(t_j)|\,\mathrm{d}t_j$, whose integral over the ordered simplex is $M_q(x)^m/m!$. This proves (4.3). Differentiation removes the outermost difference factor and gives (4.4). ☐

Theorem 4.2 (Entire dependence on the spectral parameter)**.** *For fixed $q \in \mathcal{Q}_R$, $\nu \in \mathbb{C}$, and $x \in [0,R]$, the functions*

$$k \longmapsto \phi_\nu(x,k;q), \qquad k \longmapsto \phi_\nu'(x,k;q)$$

are even entire functions. The series

$$\phi_\nu(\,\cdot\,,k;q) = \sum_{m=0}^\infty (-k^2)^m \mathcal{V}_q^m(1+\nu x) \tag{4.5}$$

and its once-differentiated series converge uniformly for $x \in [0,R]$ and k in compact subsets of $\mathbb{C}$. Moreover,

$$\|\phi_\nu(\,\cdot\,,k;q)\|_\infty \leq (1+|\nu|R)\exp\big(|k|^2 R M_q\big). \tag{4.6}$$

Proof. Lemma 4.1 majorizes (4.5) by the exponential series in (4.6). The differentiated series is locally uniformly convergent by (4.4). Consequently the sum belongs to $C^1([0,R])$, satisfies (4.1), and is unique by the usual Volterra iteration argument. Uniform convergence on compact subsets of $\mathbb{C}$ proves entire dependence. Only even powers of k occur. ☐

The weak-coupling parameter is not merely formal.

Theorem 4.3 (Joint entire dependence). *Fix $V \in \mathcal{Q}_R$, $\nu \in \mathbb{C}$, and $x \in [0, R]$. For $q = \varepsilon V$, the maps*

$$(k, \varepsilon) \longmapsto \phi_\nu(x, k; \varepsilon V), \qquad (k, \varepsilon) \longmapsto \phi'_\nu(x, k; \varepsilon V)$$

are entire on $\mathbb{C}^2$. The assertion is locally uniform in x, and hence the endpoint data $F_\nu(k; \varepsilon V)$ and $G_\nu(k; \varepsilon V)$ are entire on $\mathbb{C}^2$.

Proof. For $q = \varepsilon V$, each term $\mathcal{V}^m_{\varepsilon V}(1 + \nu x)$ in (4.5) is a polynomial in ε of degree at most m. On a compact subset on which $|\varepsilon| \le E$,

$$M_{\varepsilon V} \le R + E\|V\|_{L^1(0,R)}.$$

The estimates in Lemma 4.1 therefore give locally uniform convergence in (k, ε). The Weierstrass theorem in several complex variables proves joint holomorphy, and an entire function results because the compact set was arbitrary. The same argument applies after one differentiation in x. □

Proposition 4.4 (Continuity with respect to the coefficient). *Fix compact sets $K \subset \mathbb{C}$ and $N \subset \mathbb{C}$, and $M > 0$. There is a constant $C = C(K, N, M, R)$ such that, whenever*

$$k \in K, \quad \nu \in N, \quad q_1, q_2 \in \mathcal{Q}_R, \quad \|q_j\|_{L^1} \le M,$$

one has

$$\|\phi_\nu(\,\cdot\,, k; q_1) - \phi_\nu(\,\cdot\,, k; q_2)\|_{C^1([0,R])} \le C|k|^2\|q_1 - q_2\|_{L^1(0,R)}. \tag{4.7}$$

Proof. Subtracting the two Volterra equations gives

$$\begin{aligned} \phi_{q_1} - \phi_{q_2} = & -k^2\mathcal{V}_{q_1}(\phi_{q_1} - \phi_{q_2}) \\ & - k^2\int_0^x (x - t)(q_1(t) - q_2(t))\phi_{q_2}(t)\,\mathrm{d}t. \end{aligned}$$

The inhomogeneous term is bounded in C^1 by $C_0|k|^2\|q_1 - q_2\|_1$, uniformly on the stated sets, by (4.6). Resolving the Volterra equation by its uniformly bounded Neumann series proves (4.7). □

5. The threshold as a removable singularity

Introduce the entire endpoint combinations

$$D_\nu(k; q) := \mathrm{i}kF_\nu(k; q) + G_\nu(k; q), \qquad N_\nu(k; q) := \mathrm{i}kF_\nu(k; q) - G_\nu(k; q). \tag{5.1}$$

For $k \neq 0$, formula (3.6) reads

$$S_\nu(k; q) = e^{2\mathrm{i}kR}\frac{N_\nu(k; q)}{D_\nu(k; q)}.$$

Theorem 4.2 makes this a meromorphic continuation to the complex k-plane. We now show that the origin is never a pole.

Theorem 5.1 (Analyticity at the threshold)**.** *Let $q \in \mathcal{Q}_R$ and $\nu \in \mathbb{R}$. The meromorphic boundary quotient*

$$e^{2\mathrm{i}kR}\frac{N_\nu(k;q)}{D_\nu(k;q)},$$

defined wherever its denominator is nonzero, is holomorphic in a neighborhood of $k = 0$. Its threshold value is

$$S_\nu(0;q) = \begin{cases} -1, & \nu \neq 0, \\ 1, & \nu = 0. \end{cases} \tag{5.2}$$

For $q = \varepsilon V$, this continuation is jointly holomorphic in (k, ε) near every point $(0, \varepsilon_0) \in \mathbb{C}^2$.

Proof. At $k = 0$, the regular solution is $1 + \nu x$, independently of q. Hence

$$F_\nu(0;q) = 1 + \nu R, \qquad G_\nu(0;q) = \nu. \tag{5.3}$$

If $\nu \neq 0$, then $D_\nu(0;q) = \nu \neq 0$, so the quotient is holomorphic and has value -1.

Suppose now that $\nu = 0$. Differentiating (4.1) in x gives

$$G_0(k;q) = -k^2 \int_0^R (1 + q(t))\phi_0(t,k;q)\,\mathrm{d}t. \tag{5.4}$$

Thus

$$G_0(k;q) = k^2\widehat{G}_0(k;q), \qquad \widehat{G}_0(k;q) := -\int_0^R (1 + q(t))\phi_0(t,k;q)\,\mathrm{d}t, \tag{5.5}$$

where $\widehat{G}_0$ is even and entire. Cancelling the common factor k in (3.6) yields

$$S_0(k;q) = e^{2\mathrm{i}kR}\frac{\mathrm{i}F_0(k;q) - k\widehat{G}_0(k;q)}{\mathrm{i}F_0(k;q) + k\widehat{G}_0(k;q)}. \tag{5.6}$$

The denominator equals i at the origin, and (5.2) follows. For $q = \varepsilon V$, Theorem 4.3 makes every function used above jointly entire, and the same nonvanishing arguments hold near $(0, \varepsilon_0)$. □

Corollary 5.2 (Convergent expansions and remainders)**.** *For fixed $q \in \mathcal{Q}_R$ and $\nu \in \mathbb{R}$, there are $r > 0$ and coefficients $s_n(\nu;q)$ such that*

$$S_\nu(k;q) = \sum_{n=0}^{\infty} s_n(\nu;q)k^n, \qquad |k| < r, \tag{5.7}$$

with absolute and locally uniform convergence. For every $N \geq 0$, there is $C_N > 0$ such that

$$\left| S_\nu(k;q) - \sum_{n=0}^{N} s_n(\nu;q)k^n \right| \leq C_N|k|^{N+1}, \qquad |k| \leq \frac{r}{2}. \tag{5.8}$$

If $q = \varepsilon V$, then on a sufficiently small polydisc

$$S_\nu(k;\varepsilon V) = \sum_{m,n\geq 0} s_{m,n}(\nu;V)\varepsilon^m k^n \tag{5.9}$$

with locally absolute convergence. Rectangular truncations satisfy

$$S_\nu(k;\varepsilon V) - \sum_{m=0}^{M}\sum_{n=0}^{N} s_{m,n}(\nu;V)\varepsilon^m k^n = O(|\varepsilon|^{M+1} + |k|^{N+1}) \tag{5.10}$$

on every smaller polydisc.

Proof. The assertions are standard consequences of one- and several-variable holomorphy. For completeness, choose circles of radii r and ρ contained in the domain of holomorphy. Cauchy's integral formula bounds the coefficients by

$$|s_{m,n}| \le \frac{\sup_{|k|=r,\,|\varepsilon|=\rho}|S_\nu(k;\varepsilon V)|}{\rho^m r^n}.$$

Summing the resulting geometric tails on a smaller polydisc proves (5.8) and (5.10). □

Remark 5.3. Thus the low-energy expansion is genuinely convergent; no k-dependent matching interval is required.

The threshold jump in (5.2) is already present in the free problem. It is therefore useful to remove the endpoint contribution.

Definition 5.4 (Boundary-normalized coefficient)**.** For $\nu \neq 0$, define

$$\mathscr{S}_\nu(k;q) := \frac{S_\nu(k;q)}{S_\nu^{\mathrm{free}}(k)}. \tag{5.11}$$

For $\nu = 0$, put $\mathscr{S}_0(k;q) := S_0(k;q)$.

For fixed real ν, the function $\mathscr{S}_\nu$ is holomorphic near zero and

$$\mathscr{S}_\nu(0;q) = 1. \tag{5.12}$$

This is the quotient in which the first coefficient depending on q has an invariant interpretation.

Proposition 5.5 (Coefficient constraints)**.** *Let q and ν be real, and let $s_n = s_n(\nu;q)$ be as in* (5.7)*. Then*

$$\overline{s_n} = (-1)^n s_n. \tag{5.13}$$

Thus the even coefficients are real and the odd coefficients are purely imaginary. In addition,

$$\sum_{j=0}^{n} s_j\overline{s_{n-j}} = 0, \qquad n \ge 1. \tag{5.14}$$

Proof. The identity $S_\nu(-k;q) = \overline{S_\nu(k;q)}$ holds for real k and extends to the Taylor coefficients at zero, giving (5.13). Expanding $S_\nu(k;q)\overline{S_\nu(k;q)} = 1$ for real k and comparing coefficients gives (5.14). □

6. An all-order coefficient recursion

The analyticity theorem proves the existence of the expansion. We now make it constructive. Define

$$\begin{aligned} p_0(x;\nu,q) &:= 1+\nu x, \\ p_m(x;\nu,q) &:= -\int_0^x (x-t)(1+q(t))p_{m-1}(t;\nu,q)\,\mathrm{d}t, \qquad m\geq 1. \end{aligned} \tag{6.1}$$

Then

$$\phi_\nu(x,k;q) = \sum_{m=0}^{\infty} p_m(x;\nu,q)k^{2m}. \tag{6.2}$$

Let

$$a_m(\nu;q) := p_m(R;\nu,q), \qquad b_m(\nu;q) := p_m'(R;\nu,q). \tag{6.3}$$

In particular,

$$b_0(\nu;q) = \nu, \qquad b_m(\nu;q) = -\int_0^R (1+q(t))p_{m-1}(t;\nu,q)\,\mathrm{d}t \quad (m\geq 1), \tag{6.4}$$

and

$$F_\nu(k;q) = \sum_{m=0}^{\infty} a_m(\nu;q)k^{2m}, \qquad G_\nu(k;q) = \sum_{m=0}^{\infty} b_m(\nu;q)k^{2m}. \tag{6.5}$$

Proposition 6.1 (Bounds for the regular coefficients)**.** *For $m\geq 1$,*

$$\|p_m(\,\cdot\,;\nu,q)\|_\infty \leq (1+|\nu|R)\frac{R^m M_q^m}{m!}, \tag{6.6}$$

$$\|p_m'(\,\cdot\,;\nu,q)\|_\infty \leq (1+|\nu|R)\frac{R^{m-1} M_q^m}{m!}. \tag{6.7}$$

Consequently, the series (6.2) *and* (6.5) *converge for every $k\in\mathbb{C}$.*

Proof. By (6.1),

$$p_m = (-1)^m \mathcal{V}_q^m(1+\nu x).$$

The result follows immediately from Lemma 4.1. □

To pass from a_m, b_m to the coefficients of S_ν, we use ordinary division of convergent power series. The Neumann case must be divided only after removal of the common factor k.

Theorem 6.2 (All-order scattering recursion)**.** *Let $s_n(\nu;q)$ be defined by* (5.7).

If $\nu\neq 0$, define, for $m\geq 0$,

$$\begin{aligned} d_{2m} &= b_m(\nu;q), & d_{2m+1} &= \mathrm{i}a_m(\nu;q), \\ e_{2m} &= -b_m(\nu;q), & e_{2m+1} &= \mathrm{i}a_m(\nu;q). \end{aligned} \tag{6.8}$$

If $\nu = 0$, define instead

$$\begin{aligned} d_{2m} &= \mathrm{i}a_m(0;q), & d_{2m+1} &= b_{m+1}(0;q), \\ e_{2m} &= \mathrm{i}a_m(0;q), & e_{2m+1} &= -b_{m+1}(0;q). \end{aligned} \tag{6.9}$$

In either case $d_0 \neq 0$. Set

$$c_0 := \frac{e_0}{d_0}, \qquad c_n := \frac{1}{d_0}\left(e_n - \sum_{j=1}^{n} d_j c_{n-j}\right), \quad n \geq 1. \tag{6.10}$$

Then

$$s_n(\nu; q) = \sum_{j=0}^{n} \frac{(2\mathrm{i}R)^j}{j!} c_{n-j}, \qquad n \geq 0. \tag{6.11}$$

Proof. For $\nu \neq 0$, the definitions (6.8) are exactly the coefficients of

$$D_\nu(k; q) = \sum_{n=0}^{\infty} d_n k^n, \qquad N_\nu(k; q) = \sum_{n=0}^{\infty} e_n k^n.$$

Here $d_0 = \nu$. If $N_\nu / D_\nu = \sum c_n k^n$, comparison in

$$\left(\sum_{j\geq 0} d_j k^j\right)\left(\sum_{\ell \geq 0} c_\ell k^\ell\right) = \sum_{n\geq 0} e_n k^n$$

gives (6.10).

For $\nu = 0$, $b_0(0; q) = 0$, and (5.5) permits cancellation of k. Explicitly,

$$\frac{N_0(k;q)}{D_0(k;q)} = \frac{\mathrm{i}\sum_{m\geq 0} a_m(0;q) k^{2m} - \sum_{m\geq 0} b_{m+1}(0;q) k^{2m+1}}{\mathrm{i}\sum_{m\geq 0} a_m(0;q) k^{2m} + \sum_{m\geq 0} b_{m+1}(0;q) k^{2m+1}}.$$

This is (6.9), with $d_0 = \mathrm{i}$. Finally, multiplication by the Taylor series of $e^{2\mathrm{i}kR}$ gives (6.11). □

Remark 6.3. The recursion consists of three explicit operations: Volterra integration in (6.1), division by the nonzero scalar d_0 in (6.10), and convolution with an exponential in (6.11). It avoids polynomial matching on an unbounded exterior interval and applies without smoothness of q.

7. Explicit low-energy coefficients

For $j \geq 0$, define the moments

$$Q_j(q) := \int_0^R t^j q(t)\, \mathrm{d}t. \tag{7.1}$$

When no confusion is possible, we write $Q_j = Q_j(q)$. The cubic Neumann coefficient also contains the ordered quadratic functional

$$\mathcal{J}(q) := \int_{0<s<t<R} (t-s) q(t) q(s)\, \mathrm{d}s\, \mathrm{d}t = \frac{1}{2}\int_0^R \int_0^R |t-s| q(t) q(s)\, \mathrm{d}s\, \mathrm{d}t. \tag{7.2}$$

The second identity follows by symmetry and is valid for complex-valued q as an algebraic identity, with no complex conjugation.

Theorem 7.1 (Fixed Robin and Neumann coefficients)**.** *Let $q \in \mathcal{Q}_R$.*

For fixed $\nu \neq 0$,

$$\begin{aligned} S_\nu(k;q) = &-1 + \frac{2\mathrm{i}}{\nu}k + \frac{2}{\nu^2}k^2 \\ &+ 2\mathrm{i}\left[\frac{Q_0}{\nu^2} + \frac{2Q_1}{\nu} + Q_2 - \frac{1}{\nu^3}\right]k^3 + O(k^4). \end{aligned} \tag{7.3}$$

Equivalently,

$$\mathscr{S}_\nu(k;q) = 1 - 2\mathrm{i}\left[\int_0^R q(t)(t+\nu^{-1})^2\,\mathrm{d}t\right]k^3 + O(k^4). \tag{7.4}$$

For $\nu = 0$,

$$\begin{aligned} S_0(k;q) = 1 &- 2\mathrm{i}Q_0 k - 2Q_0^2 k^2 \\ &+ 2\mathrm{i}\left[Q_2 + Q_0Q_1 + Q_0^3 + \mathcal{J}(q)\right]k^3 + O(k^4). \end{aligned} \tag{7.5}$$

All remainder estimates are for fixed q and fixed ν; they are locally uniform on bounded subsets of $\mathcal{Q}_R$, and the first is locally uniform away from $\nu = 0$.

Proof. The first Volterra coefficient is

$$p_1(x;\nu,q) = -\int_0^x (x-t)(1+q(t))(1+\nu t)\,\mathrm{d}t.$$

Consequently,

$$b_1(\nu;q) = -R - \frac{\nu R^2}{2} - Q_0 - \nu Q_1, \tag{7.6}$$

$$a_1(\nu;q) = -\frac{R^2}{2} - \frac{\nu R^3}{6} - RQ_0 + Q_1 - \nu RQ_1 + \nu Q_2. \tag{7.7}$$

Using $a_0 = 1 + \nu R$ and $b_0 = \nu$, substitute

$$F_\nu = a_0 + a_1k^2 + O(k^4), \qquad G_\nu = b_0 + b_1k^2 + O(k^4)$$

into (3.6). Division of the two power series and multiplication by

$$e^{2\mathrm{i}kR} = 1 + 2\mathrm{i}Rk - 2R^2k^2 - \frac{4}{3}\mathrm{i}R^3k^3 + O(k^4)$$

give (7.3). The terms containing only R combine into the Taylor expansion of (3.8). Dividing by that free coefficient gives (7.4).

For $\nu = 0$, put $w = 1 + q$. Then

$$a_1(0;q) = -\int_0^R (R-t)w(t)\,\mathrm{d}t = -\frac{R^2}{2} - RQ_0 + Q_1, \tag{7.8}$$

$$b_1(0;q) = -\int_0^R w(t)\,\mathrm{d}t = -(R+Q_0), \tag{7.9}$$

$$\begin{aligned} b_2(0;q) &= \int_{0<s<t<R} (t-s)w(t)w(s)\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{R^3}{6} + \frac{R^2Q_0}{2} - RQ_1 + Q_2 + \mathcal{J}(q). \end{aligned} \tag{7.10}$$

After the cancellation in (5.6), set

$$X(k) := \frac{k\widehat{G}_0(k;q)}{\mathrm{i}F_0(k;q)} = -\mathrm{i}b_1k - \mathrm{i}(b_2 - a_1b_1)k^3 + O(k^5).$$

The quotient in (5.6) equals

$$\frac{1 - X(k)}{1 + X(k)} = 1 - 2X(k) + 2X(k)^2 - 2X(k)^3 + O(k^4).$$

Multiplication by $e^{2\mathrm{i}kR}$, followed by substitution of (7.8)–(7.10), gives

$$[k^3]S_0 = 2\mathrm{i}\big(Q_2 + Q_0Q_1 + Q_0^3 + \mathcal{J}(q)\big),$$

and the lower coefficients are those in (7.5). □

Remark 7.2 (Where the coefficient first enters). For fixed $\nu \neq 0$, the coefficients of k^0, k^1, k^2 in (7.3) are independent of q; the first coefficient that can depend on q is cubic. At the Neumann endpoint, the first coefficient that can depend on q is linear. For sign-changing q, any displayed moment may vanish, so these are structural rather than unconditional nonvanishing statements.

Corollary 7.3 (Three low moments from Robin data)**.** *Let ν_1, ν_2, ν_3 be distinct nonzero real numbers. Knowledge of the cubic coefficient of the boundary-normalized function $\mathscr{S}_{\nu_j}(k;q)$ for $j = 1, 2, 3$ determines*

$$Q_0(q), \qquad Q_1(q), \qquad Q_2(q).$$

Proof. Apart from the factor $-2\mathrm{i}$, the cubic coefficient in (7.4) is

$$Q_2 + \frac{2Q_1}{\nu} + \frac{Q_0}{\nu^2}. \tag{7.11}$$

This is a quadratic polynomial in $z = \nu^{-1}$. Its values at the three distinct points $z_j = \nu_j^{-1}$ determine its three coefficients by invertibility of the Vandermonde matrix. □

Corollary 7.4 (Positive coefficients)**.** *If $q \geq 0$ almost everywhere and $q \not\equiv 0$, then*

$$\int_0^R q(t)(t + \nu^{-1})^2 \,\mathrm{d}t > 0$$

for every $\nu \neq 0$. Hence the cubic coefficient-dependent term in (7.4) *is nonzero. In the Neumann case $Q_0 > 0$, so the linear coefficient-dependent term is nonzero.*

Proof. The square can vanish at no more than one point, while a nonzero L^1 function cannot be supported on a set of measure zero. □

8. Fréchet dependence on the coefficient

We now regard $\mathcal{Q}_R$ as the complex Banach space $L^1(0,R;\mathbb{C})$, extended by zero to the half-line. The Volterra representation gives more than continuity with respect to q.

Proposition 8.1 (Banach-space holomorphy). *For fixed $k, \nu \in \mathbb{C}$, the map*

$$q \longmapsto \phi_\nu(\,\cdot\,, k; q)$$

is an entire map from $\mathcal{Q}_R$ to $C^1([0,R])$. Hence

$$q \longmapsto F_\nu(k;q), \qquad q \longmapsto G_\nu(k;q)$$

are entire scalar-valued maps. On the open set

$$\mathcal{U}_{\nu,k} := \{q \in \mathcal{Q}_R : D_\nu(k;q) \neq 0\},$$

the map $q \mapsto S_\nu(k;q)$ is Fréchet holomorphic.

Proof. Write $\mathcal{V}_q = \mathcal{V}_0 + \mathcal{W}_q$, where

$$(\mathcal{W}_q f)(x) = \int_0^x (x-t)q(t)f(t)\,\mathrm{d}t.$$

The map $q \mapsto \mathcal{W}_q$ is bounded and linear from $\mathcal{Q}_R$ to $\mathcal{L}(C([0,R]))$. Each term in the series (4.5) is a continuous polynomial in q, and Lemma 4.1 gives uniform convergence on every bounded subset of $\mathcal{Q}_R$. The sum is therefore entire as a Banach-space-valued map. The assertion for the derivative in x follows from (4.4). Evaluation at R is continuous, and the last statement follows by composition with the holomorphic quotient on the set where its denominator is nonzero. $\square$

The derivative of the quotient admits the following useful closed form in the present normalization; compare the reflection-map linearizations in [17].

Theorem 8.2 (Exact contrast derivative). *Let $q, h \in \mathcal{Q}_R$, $k \neq 0$, and suppose $D_\nu(k;q) \neq 0$. Then*

$$D_q S_\nu(k;q)[h] = \frac{2\mathrm{i}k^3 e^{2\mathrm{i}kR}}{D_\nu(k;q)^2} \int_0^R h(x)\phi_\nu(x,k;q)^2\,\mathrm{d}x. \tag{8.1}$$

If $u(\,\cdot\,,k;q)$ denotes the normalized solution in (1.4)*, then equivalently*

$$D_q S_\nu(k;q)[h] = -\frac{\mathrm{i}k}{2}\int_0^R h(x)u(x,k;q)^2\,\mathrm{d}x. \tag{8.2}$$

Proof. Let

$$\psi(x) := D_q\phi_\nu(x,k;q)[h].$$

Differentiating the differential equation and the initial data gives

$$\psi'' + k^2(1+q)\psi = -k^2 h\phi_\nu, \qquad \psi(0) = \psi'(0) = 0. \tag{8.3}$$

For

$$W(x) := \phi_\nu(x)\psi'(x) - \phi_\nu'(x)\psi(x),$$

equation (8.3) implies

$$W'(x) = -k^2 h(x)\phi_\nu(x)^2$$

almost everywhere. Since $W(0) = 0$,

$$F_\nu D_q G_\nu[h] - G_\nu D_q F_\nu[h] = -k^2 \int_0^R h(x)\phi_\nu(x)^2 \,\mathrm{d}x. \tag{8.4}$$

Let $N = N_\nu(k;q)$ and $D = D_\nu(k;q)$. Direct differentiation yields

$$\begin{aligned}(D_q N[h])D - N(D_q D[h]) &= 2\mathrm{i}k\big(G_\nu D_q F_\nu[h] - F_\nu D_q G_\nu[h]\big)\\ &= 2\mathrm{i}k^3 \int_0^R h(x)\phi_\nu(x)^2 \,\mathrm{d}x.\end{aligned}$$

Differentiating (3.6) now proves (8.1).

By (3.4), the normalized solution is

$$u(x,k;q) = \frac{2\mathrm{i}ke^{\mathrm{i}kR}}{D_\nu(k;q)}\phi_\nu(x,k;q). \tag{8.5}$$

Substitution in (8.1) gives (8.2). □

Remark 8.3. Formula (8.2) is independent of the auxiliary support bound R. It also shows that the variation is governed by the square of the normalized solution, rather than its modulus squared. The latter appears only after division by S_ν for real data.

The differential identity has an exact finite-difference counterpart. Besides giving a nonlinear version of the first variation, it identifies precisely how a change of the coefficient can be compensated by a change of the endpoint parameter.

Theorem 8.4 (Exact two-pair identity)**.** *Let $q_0, q_1 \in \mathcal{Q}_R$, let $\nu_0, \nu_1 \in \mathbb{C}$, and fix $k \neq 0$ such that $D_{\nu_j}(k;q_j) \neq 0$ for $j = 0,1$. Write*

$$\phi_j(x) := \phi_{\nu_j}(x,k;q_j), \qquad D_j := D_{\nu_j}(k;q_j).$$

Then

$$S_{\nu_1}(k;q_1) - S_{\nu_0}(k;q_0) = \frac{2\mathrm{i}ke^{2\mathrm{i}kR}}{D_1 D_0}\left[\nu_0 - \nu_1 + k^2 \int_0^R (q_1 - q_0)\phi_1\phi_0 \,\mathrm{d}x\right]. \tag{8.6}$$

In particular, for a common endpoint parameter ν,

$$S_\nu(k;q_1) - S_\nu(k;q_0) = \frac{2\mathrm{i}k^3 e^{2\mathrm{i}kR}}{D_1 D_0}\int_0^R (q_1 - q_0)\phi_1\phi_0 \,\mathrm{d}x. \tag{8.7}$$

If u_j are the corresponding exterior-normalized solutions, then

$$S_\nu(k;q_1) - S_\nu(k;q_0) = -\frac{\mathrm{i}k}{2}\int_0^R (q_1 - q_0)u_1 u_0 \,\mathrm{d}x. \tag{8.8}$$

Proof. Set $W = \phi_1\phi_0' - \phi_1'\phi_0$. The two differential equations give, almost everywhere,

$$W' = k^2(q_1 - q_0)\phi_1\phi_0, \qquad W(0) = \nu_0 - \nu_1.$$

Consequently,

$$W(R) = \nu_0 - \nu_1 + k^2 \int_0^R (q_1 - q_0)\phi_1\phi_0 \, \mathrm{d}x. \tag{8.9}$$

Cross multiplication in the two endpoint quotients gives

$$N_1 D_0 - N_0 D_1 = 2\mathrm{i}k\big(F_1 G_0 - G_1 F_0\big) = 2\mathrm{i}kW(R).$$

Substitution in (3.6) proves (8.6), and equal endpoint parameters give (8.7). Finally, use (8.5) for both regular solutions to obtain (8.8). □

Corollary 8.5 (Boundary compensation). *For real q_j, ν_j and $k > 0$,*

$$S_{\nu_1}(k; q_1) = S_{\nu_0}(k; q_0) \tag{8.10}$$

if and only if

$$\nu_1 - \nu_0 = k^2 \int_0^R (q_1 - q_0)(x)\phi_{\nu_1}(x, k; q_1)\phi_{\nu_0}(x, k; q_0) \, \mathrm{d}x. \tag{8.11}$$

Moreover,

$$\partial_\nu S_\nu(k; q) = -\frac{2\mathrm{i}ke^{2\mathrm{i}kR}}{D_\nu(k; q)^2}, \tag{8.12}$$

and the joint first variation is

$$D_{(q,\nu)} S_\nu(k; q)[h, \eta] = \partial_\nu S_\nu(k; q) \left(\eta - k^2 \int_0^R h\phi_\nu^2 \, \mathrm{d}x \right). \tag{8.13}$$

Thus the infinitesimal isoscattering directions are exactly

$$\eta = k^2 \int_0^R h(x)\phi_\nu(x, k; q)^2 \, \mathrm{d}x. \tag{8.14}$$

Proof. For real data the prefactor in (8.6) is nonzero, so (8.10) is equivalent to (8.11). Formula (8.12) follows either by differentiating the endpoint quotient, using $F_\nu \partial_\nu G_\nu - G_\nu \partial_\nu F_\nu = 1$, or by setting $q_1 = q_0$ in (8.6) and letting $\nu_1 \to \nu_0$. Combining it with (8.1) proves (8.13) and (8.14). □

Remark 8.6 (Local isoscattering graph). For fixed real $k > 0$ and a real base pair (q_0, ν_0), there is a neighborhood of q_0 in the real space $L^1(0, R)$ and a unique real-analytic function $q \mapsto \mathcal{N}(q)$, with $\mathcal{N}(q_0) = \nu_0$, such that

$$S_{\mathcal{N}(q)}(k; q) = S_{\nu_0}(k; q_0).$$

Indeed, if $S_\nu = e^{\mathrm{i}\Theta}$ locally, then

$$\partial_\nu \Theta = \frac{2k}{G_\nu^2 + k^2 F_\nu^2} > 0, \tag{8.15}$$

so the real-analytic implicit-function theorem applies. Its derivative is the exact positive-kernel functional

$$D\mathcal{N}(q_0)[h] = k^2 \int_0^R h(x)\phi_{\nu_0}(x,k;q_0)^2 \,\mathrm{d}x.$$

The finite relation satisfied by this graph is (8.11).

Corollary 8.7 (Threshold orders of the derivative)**.** *For fixed real $q, h \in \mathcal{Q}_R$,*

$$D_q S_\nu(k;q)[h] = 2\mathrm{i}k^3 \int_0^R h(t)(t+\nu^{-1})^2 \,\mathrm{d}t + O(k^4), \qquad \nu \neq 0, \tag{8.16}$$

$$D_q S_0(k;q)[h] = -2\mathrm{i}k \int_0^R h(t)\,\mathrm{d}t + O(k^2), \qquad \nu = 0. \tag{8.17}$$

The first estimate is locally uniform for ν bounded away from zero, and both are locally uniform for q, h in bounded subsets of $L^1(0,R)$.

Proof. At $k = 0$, $\phi_\nu = 1 + \nu x$ and $D_\nu(0;q) = \nu$ when $\nu \neq 0$. Formula (8.1) therefore gives (8.16). For $\nu = 0$,

$$\phi_0(x,k;q) = 1 + O(k^2), \qquad D_0(k;q) = \mathrm{i}k + O(k^2),$$

and (8.17) follows. □

8.1. **A monotonicity formula.** Suppose q, h, ν are real and $k > 0$. Along the affine path

$$q_s = q_0 + sh, \qquad 0 \le s \le 1,$$

choose a continuous real-valued phase lift $\Theta(s)$ such that

$$S_\nu(k;q_s) = e^{\mathrm{i}\Theta(s)}. \tag{8.18}$$

Such a lift exists on an interval because S_ν is continuous and has modulus one.

Theorem 8.8 (Monotonicity of the scattering phase)**.** *With the preceding notation,*

$$\Theta'(s) = -\frac{k}{2}\int_0^R h(x)|u(x,k;q_s)|^2 \,\mathrm{d}x. \tag{8.19}$$

Consequently, if $h \ge 0$, then Θ is nonincreasing. If $h \ge 0$ and $h \not\equiv 0$, then it is strictly decreasing.

Proof. For real data, the regular solution is real. In the exterior representation (3.3), reality gives $B_\nu = \overline{A_\nu}$, and hence

$$S_\nu = \frac{\overline{A_\nu}}{A_\nu}.$$

Since $u = A_\nu^{-1}\phi_\nu$,

$$\frac{u(x,k;q)^2}{S_\nu(k;q)} = \frac{\phi_\nu(x,k;q)^2}{|A_\nu(k;q)|^2} = |u(x,k;q)|^2. \tag{8.20}$$

Divide (8.2) by S_ν and use (8.18):

$$\mathrm{i}\Theta'(s) = \frac{D_q S_\nu(k;q_s)[h]}{S_\nu(k;q_s)} = -\frac{\mathrm{i}k}{2}\int_0^R h(x)|u(x,k;q_s)|^2\,\mathrm{d}x.$$

This proves (8.19). A nontrivial solution of a second-order equation cannot have a non-isolated zero unless it vanishes identically. Thus the last integral is positive for nonzero $h \geq 0$, proving strictness. □

Remark 8.9. If one writes $S_\nu = e^{2\mathrm{i}\delta_\nu}$, then

$$\delta_\nu'(s) = -\frac{k}{4}\int_0^R h(x)|u(x,k;q_s)|^2\,\mathrm{d}x.$$

The factor two is purely a convention for the phase.

8.2. **Uniform threshold stability.** The derivative identity also quantifies how the dependence on q changes as ν approaches zero.

Theorem 8.10 (Uniform stability on L^1-balls)**.** *Fix $M > 0$. There exist $k_0 > 0$ and $C > 0$, depending only on M and R, such that for real*

$$q_1, q_2 \in \mathcal{Q}_R, \qquad \|q_j\|_{L^1} \leq M,$$

real ν, and $0 < k \leq k_0$,

$$|S_\nu(k;q_1) - S_\nu(k;q_2)| \leq C\frac{k^3(1+\nu^2)}{k^2+\nu^2}\|q_1 - q_2\|_{L^1}. \tag{8.21}$$

In particular, for every $\nu_* > 0$,

$$|S_\nu(k;q_1) - S_\nu(k;q_2)| \leq C_{\nu_*}k^3\|q_1 - q_2\|_1, \qquad |\nu| \geq \nu_*, \tag{8.22}$$

$$|S_0(k;q_1) - S_0(k;q_2)| \leq Ck\|q_1 - q_2\|_1. \tag{8.23}$$

Proof. Let ϑ_q and ψ_q be the fundamental solutions with initial data

$$(\vartheta_q(0), \vartheta_q'(0)) = (1,0), \qquad (\psi_q(0), \psi_q'(0)) = (0,1).$$

The regular solution is

$$\phi_\nu = \vartheta_q + \nu\psi_q. \tag{8.24}$$

Uniformly for $\|q\|_1 \leq M$, the Volterra equation gives, as $k \to 0$,

$$\vartheta_q(R,k) = 1 + O(k^2), \qquad \vartheta_q'(R,k) = O(k^2), \tag{8.25}$$

$$\psi_q(R,k) = R + O(k^2), \qquad \psi_q'(R,k) = 1 + O(k^2). \tag{8.26}$$

It follows that

$$\|\phi_\nu(\,\cdot\,,k;q)\|_\infty \leq C_0(1+|\nu|). \tag{8.27}$$

For real data,

$$|D_\nu(k;q)|^2 = G_\nu(k;q)^2 + k^2 F_\nu(k;q)^2.$$

Using (8.24)–(8.26) and normalizing by $\sqrt{k^2+\nu^2}$, one obtains uniformly in $\nu \in \mathbb{R}$

$$|D_\nu(k;q)|^2 \geq c_0(k^2+\nu^2) \tag{8.28}$$

for sufficiently small k. Indeed, after this normalization the vector (kF_ν, G_ν) differs by $O(k)$, uniformly in ν, from a vector of Euclidean norm one.

Combining (8.1), (8.27), and (8.28) gives

$$|D_q S_\nu(k;q)[h]| \leq C\frac{k^3(1+\nu^2)}{k^2+\nu^2}\|h\|_1.$$

Integrate this estimate along $q_s = q_1 + s(q_2 - q_1)$ to obtain (8.21). The special cases follow immediately. □

Remark 8.11. For $\nu = \alpha k$ with α bounded, estimate (8.21) is of order

$$\frac{k}{1+\alpha^2}\|q_1 - q_2\|_1.$$

Thus the degeneration from cubic to linear sensitivity occurs precisely on the transition scale identified in Section 10.

The preceding estimate has the correct order, but its constant is not sharp. The exact condition number can also be identified. For $M > 0$, set

$$\mathcal{B}_M := \{q \in L^1(0,R;\mathbb{R}) : \|q\|_1 \leq M\} \tag{8.29}$$

and define the optimal global Lipschitz modulus

$$\mathfrak{L}_M(k,\nu) := \sup_{\substack{q_0,q_1\in\mathcal{B}_M \\ q_0\neq q_1}} \frac{|S_\nu(k;q_1) - S_\nu(k;q_0)|}{\|q_1 - q_0\|_1}. \tag{8.30}$$

Theorem 8.12 (Sharp L^1 condition number)**.** *For real ν, $k > 0$, and $M > 0$,*

$$\mathfrak{L}_M(k,\nu) = \sup_{q\in\mathcal{B}_M} \frac{2k^3\|\phi_\nu(\,\cdot\,,k;q)\|_\infty^2}{G_\nu(k;q)^2 + k^2F_\nu(k;q)^2}. \tag{8.31}$$

Let

$$a_{\nu,k}(x) := \cos(kx) + \frac{\nu}{k}\sin(kx), \qquad A_{\nu,k} := \|a_{\nu,k}\|_{L^\infty(0,R)}, \tag{8.32}$$

$$\Lambda_0(k,\nu) := \frac{2k^3A_{\nu,k}^2}{k^2+\nu^2}. \tag{8.33}$$

If

$$r := k^2RM < \frac{1}{2}, \qquad \eta := \frac{Mk(1+kR)^2}{1-r} < 1, \tag{8.34}$$

then, uniformly for every $\nu \in \mathbb{R}$,

$$\Lambda_0(k,\nu) \leq \mathfrak{L}_M(k,\nu) \leq \frac{\Lambda_0(k,\nu)}{(1-r)^2(1-\eta)^2}. \tag{8.35}$$

In particular,

$$\mathfrak{L}_M(k,\nu) = \Lambda_0(k,\nu)\big(1 + O_{M,R}(k)\big) \qquad (k \downarrow 0), \tag{8.36}$$

uniformly in arbitrary real choices of $\nu = \nu(k)$.

Proof. For real data, Theorem 8.2 and $|D_\nu|^2 = G_\nu^2 + k^2 F_\nu^2$ give

$$\|D_q S_\nu(k;q)\|_{L^1(0,R;\mathbb{R})\to\mathbb{C}} = \frac{2k^3\|\phi_\nu\|_\infty^2}{|D_\nu|^2}. \tag{8.37}$$

Indeed, the norm of the real functional $h \mapsto \int h\phi_\nu^2$ is $\|\phi_\nu^2\|_\infty$; normalized indicators supported in shrinking one-sided neighborhoods of a point where $|\phi_\nu|$ is maximal give the lower bound if the norm is not attained. The Banach-space fundamental theorem of calculus along line segments in the convex ball $\mathcal{B}_M$ proves that the left-hand side of (8.31) is at most the right-hand side. Conversely, at every interior point of the ball, difference quotients in arbitrary L^1 directions are admissible and give the reverse inequality. For a boundary point q, apply this conclusion to $q_n = (1-n^{-1})q$, $n \geq 2$. The map $q \mapsto (\phi_\nu, F_\nu, G_\nu)$ is continuous from L^1 to $C^1 \times \mathbb{C}^2$, and $D_\nu(k;q) \neq 0$ throughout the real ball. Hence the operator norms in (8.37) converge, which proves (8.31) on the closed ball.

Variation of constants relative to the free equation gives

$$\phi_\nu(x,k;q) = a_{\nu,k}(x) - k\int_0^x \sin\big(k(x-t)\big)q(t)\phi_\nu(t,k;q)\,\mathrm{d}t. \tag{8.38}$$

Since $|\sin(k(x-t))| \leq k(x-t)$,

$$\|\phi_\nu\|_\infty \leq \frac{A_{\nu,k}}{1-r}. \tag{8.39}$$

At the endpoint, the same equation and its derivative yield

$$|D_\nu(k;q) - e^{\mathrm{i}kR}(\nu + \mathrm{i}k)| \leq \frac{k^2M(1+kR)}{1-r}A_{\nu,k}. \tag{8.40}$$

Moreover,

$$A_{\nu,k} \leq 1 + R|\nu|, \qquad \frac{k^2A_{\nu,k}}{\sqrt{k^2+\nu^2}} \leq k(1+Rk). \tag{8.41}$$

Thus (8.40) is at most $\eta\sqrt{k^2+\nu^2}$, and consequently

$$|D_\nu(k;q)| \geq (1-\eta)\sqrt{k^2+\nu^2}. \tag{8.42}$$

Equations (8.39), (8.42), and (8.31) give the upper bound in (8.35). Taking $q = 0$ in (8.31) gives its lower bound. Finally, $r = O(k^2)$ and $\eta = O(k)$, uniformly in ν, which proves (8.36). □

The explicit free profile in (8.33) resolves all threshold regimes without separate estimates. For example, if $\nu/k \to \alpha \in \mathbb{R}$, then

$$\mathfrak{L}_M(k,\nu) \sim \frac{2k}{1+\alpha^2}. \tag{8.43}$$

For fixed $\nu \neq 0$,

$$\mathfrak{L}_M(k,\nu) \sim \frac{2A_\nu^2}{\nu^2}k^3, \qquad A_\nu := \max\{1, |1+\nu R|\}, \tag{8.44}$$

whereas

$$\mathfrak{L}_M(k,0) = 2k + O(k^3). \tag{8.45}$$

Indeed, uniformly on $\mathcal{B}_M$, $\phi_0 = 1 + O(k^2)$, $F_0 = 1 + O(k^2)$, and $G_0 = O(k^2)$; substitution in (8.31) gives $2k(1 + O(k^2))$. If $|\nu| \to 0$ and $k/|\nu| \to 0$, then

$$\mathfrak{L}_M(k,\nu) \sim \frac{2k^3}{\nu^2}. \tag{8.46}$$

The first correction on the transition scale detects both the support endpoint and the size of the admissible coefficient ball.

Corollary 8.13 (Two-term sharp crossover)**.** *Let $\alpha, \beta \in \mathbb{R}$ and write $\alpha_+ := \max\{\alpha, 0\}$. Then*

$$\mathfrak{L}_M(k, \alpha k + \beta k^2) = \frac{2k}{1+\alpha^2}\left[1 + k\left(2R\alpha_+ + \frac{2(|\alpha|M - \alpha\beta)}{1+\alpha^2}\right) + O(k^2)\right]. \tag{8.47}$$

The remainder is uniform for α, β in compact sets, with constants depending only on M, R and the chosen compact sets. In particular,

$$\mathfrak{L}_M(k, \beta k^2) = 2k + O(k^3). \tag{8.48}$$

Proof. Uniformly for $q \in \mathcal{B}_M$ and for α, β in fixed compact sets, the endpoint expansions give

$$\begin{aligned} F_{\alpha k + \beta k^2}(k; q) &= 1 + \alpha R k + O(k^2), \\ G_{\alpha k + \beta k^2}(k; q) &= \alpha k + (\beta - R - Q_0(q))k^2 + O(k^3). \end{aligned}$$

It follows that

$$|D_{\alpha k + \beta k^2}(k; q)|^2 = (1+\alpha^2)k^2 + 2\alpha(\beta - Q_0(q))k^3 + O(k^4). \tag{8.49}$$

Uniformly in x, the Volterra equation gives $\phi_{\alpha k + \beta k^2}(x, k; q) = 1 + \alpha k x + O(k^2)$. For small k this profile is positive, and $\|1 + \alpha k \cdot\|_\infty = 1 + R\alpha_+ k$. Hence

$$\|\phi_{\alpha k + \beta k^2}(\,\cdot\,, k; q)\|_\infty^2 = 1 + 2R\alpha_+ k + O(k^2). \tag{8.50}$$

Substitute (8.49) and (8.50) into (8.31). The remainder after division is uniform in q, so the supremum may be taken at this order. Since

$$\sup_{q \in \mathcal{B}_M} \alpha Q_0(q) = |\alpha| M$$

formula (8.47) follows. Setting $\alpha = 0$ gives (8.48). ☐

9. Weak-coupling expansion and Born recursion

We now specialize to the pencil $q = \varepsilon V$, with $V \in \mathcal{Q}_R$ fixed. In contrast to the low-energy recursion of Section 6, the expansion in this section is organized by the number of occurrences of V. Both expansions are convergent near the origin, and hence may be combined without an interchange-of-limits argument.

Let

$$\phi_\nu^{[0]}(x,k) := \cos(kx) + \frac{\nu}{k}\sin(kx), \tag{9.1}$$

where the expression is continued at $k = 0$, and introduce the Volterra operator

$$(\mathcal{K}_{k,V} f)(x) := -k\int_0^x \sin\big(k(x-t)\big)V(t)f(t)\,\mathrm{d}t. \tag{9.2}$$

Variation of constants relative to the free equation gives

$$\phi_\nu(x,k;\varepsilon V) = \phi_\nu^{[0]}(x,k) + \varepsilon\mathcal{K}_{k,V}\phi_\nu(x,k;\varepsilon V). \tag{9.3}$$

Proposition 9.1 (Born series for the regular solution)**.** *For every $(k,\varepsilon) \in \mathbb{C}^2$,*

$$\phi_\nu(x,k;\varepsilon V) = \sum_{m=0}^{\infty} \varepsilon^m \phi_\nu^{[m]}(x,k), \qquad \phi_\nu^{[m]} = \mathcal{K}_{k,V}^m \phi_\nu^{[0]}. \tag{9.4}$$

The series and its first derivative in x converge locally uniformly on $[0,R]\times\mathbb{C}^2$. Each coefficient is an even entire function of k, and, for $m \geq 1$,

$$(\phi_\nu^{[m]})'' + k^2\phi_\nu^{[m]} = -k^2 V\phi_\nu^{[m-1]}, \qquad \phi_\nu^{[m]}(0,k) = (\phi_\nu^{[m]})'(0,k) = 0. \tag{9.5}$$

Proof. Equation (9.3) follows by applying the free Green kernel $k^{-1}\sin(k(x-t))$ to the inhomogeneity $-\varepsilon k^2 V\phi_\nu$. Iteration gives (9.4). On compact k-sets the kernel of $\mathcal{K}_{k,V}$ is bounded by $C|V(t)|$, and integration over the ordered simplex supplies the factor $1/m!$. This proves locally uniform convergence for every $\varepsilon \in \mathbb{C}$. The differentiated kernel has the same property. Since $k\sin(k(x-t))$ and (9.1) are even entire functions of k, so is every term. Differentiating twice gives (9.5). ☐

Put

$$F_m(k) := \phi_\nu^{[m]}(R,k), \qquad G_m(k) := (\phi_\nu^{[m]})'(R,k), \tag{9.6}$$

and define

$$d_m(k) := \mathrm{i}kF_m(k) + G_m(k), \qquad e_m(k) := \mathrm{i}kF_m(k) - G_m(k). \tag{9.7}$$

Thus $D_\nu(k;\varepsilon V) = \sum d_m(k)\varepsilon^m$, and similarly for N_ν.

Theorem 9.2 (All-order Born recursion)**.** *Fix $k \neq 0$ with $\nu + \mathrm{i}k \neq 0$. Define*

$$\gamma_0(k) := \frac{e_0(k)}{d_0(k)}, \qquad \gamma_m(k) := \frac{1}{d_0(k)}\left(e_m(k) - \sum_{j=1}^{m} d_j(k)\gamma_{m-j}(k)\right) \tag{9.8}$$

for $m \geq 1$. Then, for ε in a neighborhood of zero,

$$S_\nu(k;\varepsilon V) = e^{2\mathrm{i}kR}\sum_{m=0}^{\infty}\gamma_m(k)\varepsilon^m. \tag{9.9}$$

Set

$$\rho_k := \inf\big\{|\varepsilon| : D_\nu(k;\varepsilon V) = 0\big\}, \qquad \inf\varnothing := +\infty.$$

The radius of convergence in ε is exactly ρ_k. In particular, $e^{2\mathrm{i}kR}\gamma_0 = S_\nu^{\mathrm{free}}$.

Proof. The free denominator is $d_0 = (\nu + \mathrm{i}k)e^{\mathrm{i}kR}$, so the hypothesis gives $d_0 \neq 0$. The recursion is ordinary division of the two convergent series $\sum e_m \varepsilon^m$ and $\sum d_m \varepsilon^m$. The assertion concerning the radius follows because numerator and denominator cannot vanish simultaneously: otherwise both endpoint data would vanish and uniqueness for the Cauchy problem at R would fail. ☐

The linear term has a closed form which is more informative than the recursive expression.

Theorem 9.3 (Exact first Born coefficient)**.** *For real $k \neq 0$ and $\nu \in \mathbb{R}$,*

$$S_\nu(k; \varepsilon V) = S_\nu^{\mathrm{free}}(k) + \varepsilon \mathcal{B}_\nu(k; V) + O(\varepsilon^2), \tag{9.10}$$

where

$$\mathcal{B}_\nu(k; V) = \frac{2\mathrm{i}k^3}{(\nu + \mathrm{i}k)^2} \int_0^R V(x) \left(\cos(kx) + \frac{\nu}{k}\sin(kx)\right)^2 \mathrm{d}x. \tag{9.11}$$

For $\nu \neq 0$, its boundary-normalized form is

$$\frac{\mathcal{B}_\nu(k; V)}{S_\nu^{\mathrm{free}}(k)} = -\frac{2\mathrm{i}k^3}{k^2 + \nu^2} \int_0^R V(x) \left(\cos(kx) + \frac{\nu}{k}\sin(kx)\right)^2 \mathrm{d}x. \tag{9.12}$$

Proof. Apply Theorem 8.2 at $q = 0$. Equations (3.9) and (3.10) cancel the two exponential factors and give (9.11). Division by (3.8) gives (9.12). ☐

Let

$$\mu_j := \int_0^R x^j V(x)\, \mathrm{d}x. \tag{9.13}$$

The threshold behavior of the first Born coefficient is

$$\mathcal{B}_\nu(k; V) = 2\mathrm{i}k^3 \int_0^R V(x)(x + \nu^{-1})^2\, \mathrm{d}x + O(k^4), \qquad \nu \neq 0, \tag{9.14}$$

$$\mathcal{B}_0(k; V) = -2\mathrm{i}k \int_0^R V(x) \cos^2(kx)\, \mathrm{d}x = -2\mathrm{i}\mu_0 k + 2\mathrm{i}\mu_2 k^3 + O(k^5). \tag{9.15}$$

The following statement records the nonlinear remainder with its sharp threshold order.

Proposition 9.4 (Joint weak-coupling remainders)**.** *Fix $V \in \mathcal{Q}_R$. The estimates below are for real k, ν; the parameter ε may be real or complex in a sufficiently small disk.*

(1) *For every $\nu_* > 0$, locally uniformly for real $|\nu| \geq \nu_*$ and small (k, ε),*

$$S_\nu(k; \varepsilon V) = S_\nu^{\mathrm{free}}(k) + \varepsilon \mathcal{B}_\nu(k; V) + O(\varepsilon^2 k^5). \tag{9.16}$$

In particular,

$$\begin{aligned} S_\nu(k; \varepsilon V) = S_\nu^{\mathrm{free}}(k) &+ 2\mathrm{i}\varepsilon k^3 \int_0^R V(x)(x + \nu^{-1})^2\, \mathrm{d}x \\ &+ O(|\varepsilon||k|^4 + |\varepsilon|^2 |k|^5). \end{aligned} \tag{9.17}$$

(2) *For each $A > 0$, locally uniformly when $|\nu| \le A|k|$,*

$$S_\nu(k;\varepsilon V) = S_\nu^{\mathrm{free}}(k) + \varepsilon \mathcal{B}_\nu(k;V) + O(\varepsilon^2 k^2). \tag{9.18}$$

At $\nu = 0$, the expansion through cubic order is

$$\begin{aligned} S_0(k;\varepsilon V) &= 1 - 2\mathrm{i}\varepsilon\mu_0 k - 2\varepsilon^2\mu_0^2 k^2 \\ &\quad + 2\mathrm{i}\left[\varepsilon\mu_2 + \varepsilon^2\big(\mu_0\mu_1 + \mathcal{J}(V)\big) + \varepsilon^3\mu_0^3\right] k^3 + O(k^4), \end{aligned} \tag{9.19}$$

uniformly for ε in compact sets.

Proof. Let $\psi_h = D_q\phi_\nu[h]$. The differentiated Volterra equation and the estimates of Section 4 give, on bounded L^1-sets,

$$\|\psi_h\|_{C^1([0,R])} \le Ck^2(1+|\nu|)\|h\|_1, \qquad |D_q D_\nu(k;q)[h]| \le Ck^2(1+|\nu|)\|h\|_1. \tag{9.20}$$

Differentiate (8.1) once more with respect to q. When $|\nu| \ge \nu_*$, the endpoint estimates give

$$\|\phi_\nu\|_\infty = O(1+|\nu|), \qquad |D_\nu| \ge c|\nu|.$$

Together with (9.20), these estimates show directly in both terms of the second derivative that $D_q^2 S_\nu = O(k^5)$, uniformly on the unbounded set $|\nu| \ge \nu_*$. When $|\nu| \le A|k|$, one has $|D_\nu| \asymp |k|$, uniformly for sufficiently small real k, and the same calculation gives $D_q^2 S_\nu = O(k^2)$. Taylor's formula along $q = s\varepsilon V$ proves (9.16) and (9.18). Formula (9.17) follows from (9.14), while (9.19) is (7.5) with $q = \varepsilon V$. ☐

Remark 9.5. The term $-2\varepsilon^2\mu_0^2 k^2$ in (9.19) shows that the transition-scale remainder $O(\varepsilon^2 k^2)$ is attained in general. For fixed nonzero Robin data, the preceding argument yields the higher-order bound $O(\varepsilon^2 k^5)$.

10. The Robin–Neumann transition

The fixed-ν expansions of Theorem 7.1 are not uniform as $\nu \to 0$. The correct transition variable is ν/k. We first record the endpoint estimates responsible for this fact.

Lemma 10.1 (Endpoint data on the transition scale)**.** *On bounded subsets of $\mathcal{Q}_R$, as $k \to 0$,*

$$F_\nu(k;q) = 1 + \nu R + O\big(k^2(1+|\nu|)\big), \tag{10.1}$$

$$G_\nu(k;q) = \nu - k^2\left(R + Q_0 + \nu\left(\frac{R^2}{2} + Q_1\right)\right) + O\big(k^4(1+|\nu|)\big). \tag{10.2}$$

The estimates are uniform for real or complex ν.

Proof. The zeroth Volterra coefficient is $1 + \nu x$. Formula (7.6) gives the coefficient of k^2 in G_ν, and Proposition 6.1 bounds the remaining series. The estimate for F_ν follows in the same way. ☐

Theorem 10.2 (Transition law)**.** *Let $q \in \mathcal{Q}_R$ be fixed, let $k > 0$ tend to zero, and let $\nu = \nu(k) \in \mathbb{R}$.*

(1) *If* $\nu(k)/k \to \alpha \in \mathbb{R}$, *then*

$$S_{\nu(k)}(k;q) \longrightarrow \frac{\mathrm{i}-\alpha}{\mathrm{i}+\alpha}. \tag{10.3}$$

Relative to the free coefficient at the same value of $\nu(k)$,

$$S_{\nu(k)}(k;q) = S^{\mathrm{free}}_{\nu(k)}(k) + \frac{2\mathrm{i}Q_0}{(\alpha+\mathrm{i})^2}k + o(k). \tag{10.4}$$

(2) *If* $|\nu(k)|/k \to \infty$, *then*

$$S_{\nu(k)}(k;q) \longrightarrow -1. \tag{10.5}$$

(3) *If*

$$\nu(k) = \alpha k + \beta k^2 + o(k^2), \tag{10.6}$$

then

$$S_{\nu(k)}(k;q) = \frac{\mathrm{i}-\alpha}{\mathrm{i}+\alpha} + \frac{2\mathrm{i}(Q_0-\beta)}{(\alpha+\mathrm{i})^2}k + o(k). \tag{10.7}$$

All assertions are locally uniform for q *in bounded subsets of* $\mathcal{Q}_R$, *and for* α, β *in compact subsets of* $\mathbb{R}$.

Proof. If $\nu/k \to \alpha$, Lemma 10.1 gives

$$D_\nu(k;q) = k(\alpha+\mathrm{i}) + o(k), \qquad N_\nu(k;q) = k(-\alpha+\mathrm{i}) + o(k).$$

Formula (3.6) proves (10.3). Retaining the $-k^2Q_0$ term in (10.2) and subtracting the exact free quotient proves (10.4). If $|\nu|/k \to \infty$, divide numerator and denominator by ν and use (10.1)–(10.2); this gives (10.5). Finally,

$$\frac{\mathrm{i}-(\alpha+\beta k+o(k))}{\mathrm{i}+(\alpha+\beta k+o(k))} = \frac{\mathrm{i}-\alpha}{\mathrm{i}+\alpha} - \frac{2\mathrm{i}\beta}{(\alpha+\mathrm{i})^2}k + o(k).$$

Combining this with (10.4) proves (10.7). $\square$

For the exact scaling $\nu = \alpha k$, one may continue one order further. This also displays the first nonlinear weak-coupling term in the transition regime.

Proposition 10.3 (Second-order transition expansion)**.** *Let* $k > 0$, $\alpha \in \mathbb{R}$, $q = \varepsilon V$, *and* $\nu = \alpha k$. *Define* μ_j *by* (9.13). *Uniformly for* α *and* ε *in compact sets,*

$$\begin{aligned} S_{\alpha k}(k;\varepsilon V) = {} & \frac{\mathrm{i}-\alpha}{\mathrm{i}+\alpha} + \frac{2\mathrm{i}\varepsilon\mu_0}{(\alpha+\mathrm{i})^2}k \\ & + \left[\frac{4\mathrm{i}\alpha\varepsilon\mu_1}{(\alpha+\mathrm{i})^2} + \frac{2\mathrm{i}\varepsilon^2\mu_0^2}{(\alpha+\mathrm{i})^3}\right]k^2 + O(k^3). \end{aligned} \tag{10.8}$$

Proof. With $Q_j = \varepsilon\mu_j$, the Volterra recursion gives

$$F_{\alpha k}(k;q) = 1 + \alpha k R - k^2\left(\frac{R^2}{2} + RQ_0 - Q_1\right) + O(k^3),$$

$$G_{\alpha k}(k;q) = \alpha k - k^2(R+Q_0) - \alpha k^3\left(\frac{R^2}{2} + Q_1\right) + O(k^4).$$

Substitution in (3.6) and power-series division give (10.8); all terms depending only on R cancel. □

Remark 10.4 (Relation to the bounded-coefficient literature). For bounded, piecewise-continuous profiles, Proposition 10.3 is equivalent, after reversal of the incoming-wave convention and translation of the Robin parameters, to the half-line expansion in [13, Eq. (61)]. It is retained here both to fix the present normalization and to provide the bounded-L^1 baseline for Theorems 8.12 and 10.6. The latter theorem permits $\|q_k\|_1 \asymp k^{-1}$, which is outside that bounded coefficient regime.

Corollary 10.5 (Noncommuting threshold limits). *For every fixed $q \in \mathcal{Q}_R$,*

$$\lim_{\nu \to 0,\, \nu \neq 0} \lim_{k \to 0} S_\nu(k; q) = -1, \qquad \lim_{k \to 0} S_0(k; q) = 1.$$

More generally, every point of the unit circle other than -1 is obtained from (10.3) *by a unique $\alpha \in \mathbb{R}$, while -1 is the limit $|\alpha| \to \infty$.*

10.1. **Singular contrast and critical exponents.** The preceding transition keeps the coefficient in a bounded subset of L^1. A different normal form emerges if the coefficient itself depends on k. Although $\|q_k\|_1$ may then diverge, the differential coefficient $k^2 q_k$ still tends to zero when $q_k = O(k^{-1})$. The boundary term and the integrated coefficient enter at the same order, and every power of the weak-coupling series must be resummed.

Theorem 10.6 (Critical-contrast normal form). *Let $V, W \in L^1(0, R; \mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$. Suppose, as $k \downarrow 0$, that*

$$q_k = k^{-1} V + W + r_k, \qquad \|r_k\|_1 \longrightarrow 0, \tag{10.9}$$

$$\nu(k) = \alpha k + \beta k^2 + \rho_k, \qquad \frac{|\rho_k|}{k^2} \longrightarrow 0. \tag{10.10}$$

Define

$$a := \alpha - Q_0(V), \tag{10.11}$$

$$\Gamma(V, W; \alpha, \beta) := Q_0(W) - \beta + \big(2\alpha - Q_0(V)\big) Q_1(V) - \mathcal{J}(V). \tag{10.12}$$

Then

$$S_{\nu(k)}(k; q_k) = \frac{\mathrm{i} - a}{\mathrm{i} + a} + \frac{2\mathrm{i}\Gamma(V, W; \alpha, \beta)}{(a + \mathrm{i})^2} k + o(k). \tag{10.13}$$

More precisely, if

$$\eta(k) := \|r_k\|_1 + \frac{|\rho_k|}{k^2}, \tag{10.14}$$

then the last remainder is

$$O\big(k^2 + k\eta(k)\big). \tag{10.15}$$

The implicit constant depends only on R and bounds for V, W, α, β, provided k is sufficiently small and $\eta(k) \leq 1$. It is therefore uniform for families in which $\eta(k) \to 0$ uniformly.

Proof. For $f, g \in L^1(0,R) \times C[0,R]$, write

$$(\mathcal{T}_f g)(x) := \int_0^x (x-t) f(t) g(t)\,\mathrm{d}t. \tag{10.16}$$

Substitution of (10.9) and (10.10) in the Volterra equation gives, in $C^1[0,R]$,

$$\begin{aligned}\phi_{\nu(k)}(\,\cdot\,,k;q_k) = 1 + \alpha k x + \beta k^2 x - k\mathcal{T}_V \phi_{\nu(k)} - k^2 \mathcal{T}_{1+W}\phi_{\nu(k)} \\ + O\big(k^2\eta(k)\big).\end{aligned} \tag{10.17}$$

On bounded profile sets, once $\eta(k) \le 1$, the Volterra operator on the right has norm $O(k)$ from $C[0,R]$ to $C^1[0,R]$. Hence $\|\phi_{\nu(k)}\|_{C^1} \le C$. The omitted terms $\rho_k x - k^2 \mathcal{T}_{r_k}\phi_{\nu(k)}$ are therefore $O_{C^1}(k^2\eta(k))$. A second iteration yields

$$\phi_{\nu(k)}(x,k;q_k) = 1 + k f_1(x) + k^2 f_2(x) + O\big(k^3 + k^2\eta(k)\big), \tag{10.18}$$

where

$$f_1 = \alpha x - \mathcal{T}_V 1, \qquad f_2 = \beta x - \mathcal{T}_V f_1 - \mathcal{T}_{1+W} 1. \tag{10.19}$$

If $\psi = 1 + k f_1 + k^2 f_2$, direct substitution shows that its residual in (10.17) is $O_{C^1}(k^3 + k^2\eta(k))$. The Volterra resolvent is uniformly bounded on the parameter sets in the statement, so the same estimate holds for $\phi_{\nu(k)} - \psi$. At $x = R$,

$$f_1(R) = R\big(\alpha - Q_0(V)\big) + Q_1(V) = Ra + Q_1(V), \tag{10.20}$$

$$f_1'(R) = \alpha - Q_0(V) = a, \tag{10.21}$$

$$f_2'(R) = \beta - R - Q_0(W) - \alpha Q_1(V) + \mathcal{J}(V). \tag{10.22}$$

Consequently, with $f := Ra + Q_1(V)$ and $b := f_2'(R)$,

$$F_{\nu(k)}(k;q_k) = 1 + kf + O\big(k^2 + k^2\eta(k)\big), \tag{10.23}$$

$$G_{\nu(k)}(k;q_k) = ka + k^2 b + O\big(k^3 + k^2\eta(k)\big). \tag{10.24}$$

Power-series division in the endpoint quotient gives

$$\frac{\mathrm{i}kF - G}{\mathrm{i}kF + G} = \frac{\mathrm{i} - a}{\mathrm{i} + a} + \frac{2\mathrm{i}(af - b)}{(a+\mathrm{i})^2} k + O\big(k^2 + k\eta(k)\big). \tag{10.25}$$

Multiplication by $e^{2\mathrm{i}kR} = 1 + 2\mathrm{i}kR + O(k^2)$ replaces $af - b$ by

$$\begin{aligned} af - b - R(1 + a^2) &= Q_0(W) - \beta + \big(2\alpha - Q_0(V)\big)Q_1(V) - \mathcal{J}(V) \\ &= \Gamma(V, W; \alpha, \beta).\end{aligned}$$

This proves (10.13) and (10.15). □

For a single profile, the theorem takes a particularly transparent form. Let $\lambda, \sigma \in \mathbb{R}$. With $\mu_j = Q_j(V)$, if

$$q_k = \left(\frac{\lambda}{k} + \sigma\right) V + r_k, \qquad \nu(k) = \alpha k + \beta k^2 + \rho_k, \tag{10.26}$$

under the remainder hypotheses of Theorem 10.6, then (10.13) holds with

$$a = \alpha - \lambda\mu_0, \qquad \Gamma = \sigma\mu_0 - \beta + \lambda(2\alpha - \lambda\mu_0)\mu_1 - \lambda^2 \mathcal{J}(V). \tag{10.27}$$

Thus the first correction retains the ordering information in $\mathcal{J}(V)$; it is not determined by the total mass alone.

The next result shows that k^{-1} is the generic critical power, but that cancellation of the mean shifts the critical power to $k^{-3/2}$. Put

$$\mathcal{C}(z) := \frac{\mathrm{i} - z}{\mathrm{i} + z}, \qquad z \in \mathbb{C} \setminus \{-\mathrm{i}\}. \tag{10.28}$$

Theorem 10.7 (Critical-exponent trichotomy)**.** *Let $V \in L^1(0, R; \mathbb{R})$, let*

$$q_k = k^{-\gamma} V, \qquad \nu(k) = \alpha k, \qquad 0 \le \gamma < 2. \tag{10.29}$$

If $Q_0(V) \neq 0$, then

$$S_{\alpha k}(k; k^{-\gamma} V) \longrightarrow \begin{cases} \mathcal{C}(\alpha), & 0 \le \gamma < 1, \\ \mathcal{C}\big(\alpha - Q_0(V)\big), & \gamma = 1, \\ -1, & 1 < \gamma < 2. \end{cases} \tag{10.30}$$

If $Q_0(V) = 0$ and $V \not\equiv 0$, then

$$S_{\alpha k}(k; k^{-\gamma} V) \longrightarrow \begin{cases} \mathcal{C}(\alpha), & 0 \le \gamma < \frac{3}{2}, \\ \mathcal{C}\big(\alpha + \mathcal{J}(V)\big), & \gamma = \frac{3}{2}, \\ -1, & \frac{3}{2} < \gamma < 2. \end{cases} \tag{10.31}$$

At the mean-zero critical power, the convergence rate is $O(k^{1/2})$.

Proof. Set $\delta := k^{2-\gamma}$. A Volterra iteration, now retaining the first two powers of δ, gives uniformly on bounded real L^1-sets

$$F_{\alpha k}(k; k^{-\gamma} V) = 1 + O(k + \delta), \tag{10.32}$$

$$G_{\alpha k}(k; k^{-\gamma} V) = \alpha k - \delta Q_0(V) + \delta^2 \mathcal{J}(V) + O\big(k^2 + k\delta + \delta^3\big). \tag{10.33}$$

For completeness, the first iteration is

$$\phi_{\alpha k} = 1 - \delta \mathcal{T}_V 1 + O(k + \delta^2) \quad \text{in } C[0, R]. \tag{10.34}$$

Since $\int_0^R V \mathcal{T}_V 1 = \mathcal{J}(V)$, equation (10.34) implies

$$\int_0^R V(t) \phi_{\alpha k}(t)\, \mathrm{d}t = Q_0(V) - \delta \mathcal{J}(V) + O(k + \delta^2),$$

$$\int_0^R \phi_{\alpha k}(t)\, \mathrm{d}t = O(1).$$

The exact endpoint identity

$$G_{\alpha k} = \alpha k - k^2 \int_0^R \phi_{\alpha k}(t)\, \mathrm{d}t - \delta \int_0^R V(t) \phi_{\alpha k}(t)\, \mathrm{d}t \tag{10.35}$$

now proves (10.33); the value estimate follows from the undifferentiated Volterra equation.

If $Q_0(V) \neq 0$, comparison of δ with k in (10.33) gives the three cases in (10.30). If $Q_0(V) = 0$, the first nonzero coefficient term is instead $\delta^2 \mathcal{J}(V)$. To show that it cannot vanish for a nonzero real mean-zero profile, put

$$P(x) := \int_0^x V(t)\,\mathrm{d}t. \tag{10.36}$$

Then $P(0) = P(R) = 0$, and integration by parts in (7.2) gives

$$\mathcal{J}(V) = -\int_0^R P(x)^2\,\mathrm{d}x < 0. \tag{10.37}$$

Comparing δ^2 with k proves (10.31). At $\gamma = 3/2$, $\delta = k^{1/2}$, so

$$G_{\alpha k} = k\big(\alpha + \mathcal{J}(V)\big) + O(k^{3/2}), \qquad F_{\alpha k} = 1 + O(k^{1/2}),$$

which also proves the stated rate. In every supercritical case in the indicated range, the nonzero real term in G dominates ikF, so the quotient tends to -1. $\square$

Example 10.8 (Balanced two-step profile). Let

$$V(x) = \begin{cases} 1, & 0 < x < R/2, \\ -1, & R/2 < x < R. \end{cases}$$

Then $Q_0(V) = 0$, while its primitive is $P(x) = x$ on $[0, R/2]$ and $P(x) = R - x$ on $[R/2, R]$. Hence

$$\mathcal{J}(V) = -2\int_0^{R/2} x^2\,\mathrm{d}x = -\frac{R^3}{12}.$$

At the critical power $q_k = k^{-3/2}V$,

$$S_{\alpha k}(k; k^{-3/2}V) \longrightarrow \mathcal{C}\left(\alpha - \frac{R^3}{12}\right). \tag{10.38}$$

Thus a coefficient with vanishing mean still produces a nontrivial renormalization of the boundary parameter.

Remark 10.9. The restriction $\gamma < 2$ is structural. At $\gamma = 2$, the limiting equation is $u'' + Vu = 0$, and a zero-energy transfer-matrix resonance may change the limit. For $\gamma > 2$, sign-changing profiles need not have a universal threshold limit. Thus the supercritical conclusions in Theorem 10.7 are not asserted beyond their stated range.

For mean-zero profiles, the negative form (10.37) is analogous to the second-order mechanism in weak-coupling threshold analysis [14]; the present conclusion concerns a half-line boundary quotient under simultaneous coefficient and endpoint scaling.

The critical limit also determines the leading threshold order of every Born coefficient at once.

Corollary 10.10 (Resummed leading Born hierarchy). *Fix $\alpha \in \mathbb{R}$ and $V \in L^1(0, R; \mathbb{R})$, and write*

$$S_{\alpha k}(k; \varepsilon V) = \sum_{m=0}^{\infty} b_m(k) \varepsilon^m \tag{10.39}$$

near $\varepsilon = 0$. Then, for every $m \geq 1$,

$$b_m(k) = \frac{2\mathrm{i} Q_0(V)^m}{(\alpha + \mathrm{i})^{m+1}} k^m + O_m(k^{m+1}). \tag{10.40}$$

Proof. Set $\varepsilon = \lambda / k$, and choose $r > 0$ so that

$$\inf_{|\lambda| \leq r} |\alpha + \mathrm{i} - \lambda Q_0(V)| > 0.$$

Put

$$\mathcal{F}_k(\lambda) := S_{\alpha k}\left(k; \frac{\lambda}{k} V\right).$$

After complexifying the coefficient, the Volterra equation and endpoint quotient give

$$\sup_{|\lambda| \leq r} |\mathcal{F}_k(\lambda) - \mathcal{C}\big(\alpha - \lambda Q_0(V)\big)| \leq C_r k. \tag{10.41}$$

This is the complexified leading-order part of the proof of Theorem 10.6; the choice of r keeps the limiting endpoint denominator uniformly nonzero. Since

$$\mathcal{C}\big(\alpha - \lambda Q_0(V)\big) = -1 + \frac{2\mathrm{i}}{\alpha + \mathrm{i}} \sum_{m=0}^{\infty} \left(\frac{\lambda Q_0(V)}{\alpha + \mathrm{i}}\right)^m. \tag{10.42}$$

For $0 < r_0 < r$, the coefficient of λ^m in $\mathcal{F}_k$ is $k^{-m} b_m(k)$. Cauchy's estimate on $|\lambda| = r_0$, applied to (10.41), therefore gives

$$|k^{-m} b_m(k) - \frac{2\mathrm{i} Q_0(V)^m}{(\alpha + \mathrm{i})^{m+1}}| \leq C_{m, r_0} k.$$

Multiplication by k^m proves (10.40). □

11. Complex singularities and the convergence radius

The low-energy series is convergent, but it is generally not entire. Its obstructions are described exactly by a reduced boundary denominator. Define

$$\Delta_\nu(k; q) := \begin{cases} D_\nu(k; q), & \nu \neq 0, \\ \mathrm{i} F_0(k; q) + k \widehat{G}_0(k; q), & \nu = 0, \end{cases} \tag{11.1}$$

where $\widehat{G}_0$ is given by (5.5). Thus the common factor k has already been removed in the Neumann case. The zeros of Δ_ν are independent of the choice of support bound in the following sense: enlarging R multiplies the denominator by a nonvanishing exponential factor.

Theorem 11.1 (Meromorphic continuation and genuine poles)**.** *For every $q \in \mathcal{Q}_R$ and $\nu \in \mathbb{R}$, the threshold continuation $S_\nu(\,\cdot\,; q)$ extends meromorphically to $\mathbb{C}$. Its finite poles are precisely the zeros of $\Delta_\nu(\,\cdot\,; q)$, with equality of multiplicities. Consequently, if*

$$r_\nu(q) := \inf\{|\zeta| : \Delta_\nu(\zeta; q) = 0\}, \tag{11.2}$$

with the convention $\inf \varnothing = \infty$*, then the Taylor series of $S_\nu(k; q)$ at $k = 0$ has radius of convergence exactly $r_\nu(q)$.*

Proof. For $\nu \neq 0$, the assertion follows from

$$S_\nu(k; q) = e^{2ikR} \frac{N_\nu(k; q)}{D_\nu(k; q)}$$

and the fact that N_ν, D_ν are entire. They have no common zero: if $k \neq 0$ and $N_\nu = D_\nu = 0$, then $F_\nu = G_\nu = 0$, contrary to uniqueness for the Cauchy problem at R. At $k = 0$, the denominator equals ν.

For $\nu = 0$, use the cancelled representation (5.6). Its numerator and denominator cannot vanish together at a nonzero point, since this would give $F_0 = \widehat{G}_0 = 0$, and hence $F_0 = G_0 = 0$. At zero the reduced denominator is i. Thus every denominator zero is a nonremovable pole. The radius of a Taylor series of a meromorphic function is the distance to its nearest nonremovable singularity, which proves the final assertion. □

Remark 11.2. The zeros in Theorem 11.1 may equivalently be viewed as values of the spectral parameter for which the boundary condition admits a nonzero solution proportional to a single exterior exponential. The proof uses only the analytic boundary quotient, and no spectral interpretation of nonreal zeros is needed below.

Proposition 11.3 (Symmetry and exclusion from the real axis)**.** *Suppose q and ν are real. Then*

$$\Delta_\nu(-\bar{k}; q) = \overline{\Delta_\nu(k; q)}, \qquad \nu \neq 0, \tag{11.3}$$

$$\Delta_0(-\bar{k}; q) = -\overline{\Delta_0(k; q)}. \tag{11.4}$$

In particular, the pole set is invariant under reflection across the imaginary axis. It contains no nonzero real point.

Proof. The Taylor coefficients of F_ν, G_ν, and $\widehat{G}_0$ are real, and these functions are even in k. Substitution in (11.1) gives (11.3)–(11.4). For real $k \neq 0$,

$$|D_\nu(k; q)|^2 = G_\nu(k; q)^2 + k^2 F_\nu(k; q)^2 > 0,$$

because the endpoint data cannot both vanish. Since $D_0 = k\Delta_0$, this also excludes nonzero real zeros in the Neumann case. □

Although a global lower bound for $r_\nu(q)$ is not available on arbitrary coefficient classes, the Volterra estimates give a uniform local disk.

Proposition 11.4 (A uniform pole-free disk)**.** *Fix $M > 0$ and $\nu_* > 0$. There exists $r = r(M, R, \nu_*) > 0$ such that*

$$\Delta_\nu(k; q) \neq 0, \qquad |k| < r,$$

whenever $\|q\|_{L^1} \leq M$ and either $\nu = 0$ or $|\nu| \geq \nu_*$. The coefficient q may be complex-valued. Hence the Taylor remainders in Corollary 5.2 are uniform on every strictly smaller disk for these coefficient classes.

Proof. At the origin,

$$D_\nu(0; q) = \nu, \qquad \Delta_0(0; q) = \mathrm{i},$$

independently of q. The bounds of Lemma 4.1 imply, uniformly for $\|q\|_1 \leq M$,

$$D_\nu(k; q) = \nu + \mathrm{i}k(1 + \nu R) + O\big(k^2(1 + |\nu|)\big).$$

After division by ν, the error from 1 is at most $C|k|(1 + \nu_*^{-1})$ for $|\nu| \geq \nu_*$. Choosing r small proves nonvanishing in that case. Similarly,

$$\Delta_0(k; q) = \mathrm{i} - k\big(R + Q_0(q)\big) + O(k^2)$$

uniformly on the L^1-ball, and a second reduction of r proves the Neumann assertion. After division by ν in the Robin case, the same endpoint estimates bound the numerator uniformly and the denominator uniformly away from zero on the chosen circle. In the Neumann case, the reduced numerator and denominator are both uniformly close to i. Cauchy's formula on a smaller circle therefore gives uniform coefficient and remainder bounds. □

12. An exactly solvable compact coefficient

We test the preceding identities on a constant layer. Let

$$q(x) = q_0 \mathbf{1}_{[0,R]}(x), \qquad q_0 > -1, \qquad c := \sqrt{1 + q_0}. \tag{12.1}$$

On $[0, R]$, the regular solution is

$$\phi_\nu(x, k; q_0) = \cos(ckx) + \frac{\nu}{ck}\sin(ckx), \tag{12.2}$$

and

$$\phi_\nu'(x, k; q_0) = -ck\sin(ckx) + \nu\cos(ckx).$$

Proposition 12.1 (Closed form for the constant layer)**.** *For $k \neq 0$,*

$$S_\nu(k; q_0) = e^{2\mathrm{i}kR}\frac{(\mathrm{i}k - \nu)\cos(ckR) + (ck + \mathrm{i}\nu/c)\sin(ckR)}{(\mathrm{i}k + \nu)\cos(ckR) + (-ck + \mathrm{i}\nu/c)\sin(ckR)}. \tag{12.3}$$

For $\nu = 0$, this reduces to

$$S_0(k; q_0) = e^{2\mathrm{i}kR}\frac{\mathrm{i}\cos(ckR) + c\sin(ckR)}{\mathrm{i}\cos(ckR) - c\sin(ckR)}. \tag{12.4}$$

Proof. Substitute (12.2) and its derivative at R into (3.6), and collect the coefficients of the sine and cosine. Cancelling k when $\nu = 0$ gives (12.4). □

The relevant moments are

$$Q_j = \frac{q_0 R^{j+1}}{j+1}, \qquad \mathcal{J}(q) = \frac{q_0^2 R^3}{6}. \tag{12.5}$$

Expanding the closed form at the origin yields, for fixed $\nu \neq 0$,

$$\begin{aligned} S_\nu(k;q_0) = & -1 + \frac{2\mathrm{i}}{\nu}k + \frac{2}{\nu^2}k^2 \\ & + 2\mathrm{i}\left[\frac{q_0 R}{\nu^2} + \frac{q_0 R^2}{\nu} + \frac{q_0 R^3}{3} - \frac{1}{\nu^3}\right] k^3 + O(k^4), \end{aligned} \tag{12.6}$$

and, at the Neumann endpoint,

$$\begin{aligned} S_0(k;q_0) = 1 - 2\mathrm{i}q_0 Rk - 2q_0^2 R^2 k^2 \\ + 2\mathrm{i}R^3\left(\frac{q_0}{3} + \frac{2q_0^2}{3} + q_0^3\right) k^3 + O(k^4). \end{aligned} \tag{12.7}$$

These are exactly (7.3) and (7.5) after (12.5) is inserted. In particular, the ordered quadratic moment is necessary for the coefficient $2q_0^2R^3/3$ in (12.7).

Corollary 12.2 (Checks on the limiting regimes)**.** *The closed form* (12.3) *has the following properties.*

(1) *If $q_0 = 0$, then $S_\nu(k;0) = (\mathrm{i}k - \nu)/(\mathrm{i}k + \nu)$.*
(2) *Differentiating at $q_0 = 0$ gives*

$$\left.\frac{\partial S_\nu(k;q_0)}{\partial q_0}\right|_{q_0=0} = \mathcal{B}_\nu(k;\mathbf{1}_{[0,R]}),$$

with $\mathcal{B}_\nu$ as in (9.11)*.*
(3) *On the exact transition scale $\nu = \alpha k$,*

$$S_{\alpha k}(k;q_0) = \frac{\mathrm{i}-\alpha}{\mathrm{i}+\alpha} + \frac{2\mathrm{i}q_0 R}{(\alpha+\mathrm{i})^2}k + O(k^2).$$

Proof. The first assertion follows from elementary trigonometric factorization. The second follows either by direct differentiation or by Theorem 9.3. The last follows from Proposition 10.3. □

13. Consequences, limitations, and applications

13.1. **Low-energy moment information.** The expansions have a direct consequence for local inverse questions. For $\nu \neq 0$, set

$$A_\nu(q) := \frac{\mathrm{i}}{12}\frac{\mathrm{d}^3}{\mathrm{d}k^3}\mathscr{S}_\nu(0;q).$$

Equations (7.4) and (7.5) give

$$Q_0(q) = \frac{\mathrm{i}}{2}S_0'(0;q), \qquad A_\nu(q) = Q_2 + \frac{2Q_1}{\nu} + \frac{Q_0}{\nu^2}. \tag{13.1}$$

Thus the first Neumann derivative determines the total coefficient, whereas one boundary-normalized Robin cubic derivative determines a weighted second moment. For three distinct nonzero values ν_1, ν_2, ν_3, the system

$$\begin{pmatrix} 1 & 2/\nu_1 & 1/\nu_1^2 \\ 1 & 2/\nu_2 & 1/\nu_2^2 \\ 1 & 2/\nu_3 & 1/\nu_3^2 \end{pmatrix} \begin{pmatrix} Q_2 \\ Q_1 \\ Q_0 \end{pmatrix} = \begin{pmatrix} A_{\nu_1} \\ A_{\nu_2} \\ A_{\nu_3} \end{pmatrix} \tag{13.2}$$

is nonsingular. This recovers Corollary 7.3 in a form that makes the data-to-moment map explicit. No uniqueness result for the full coefficient from a finite low-energy jet is asserted.

The phase identity gives a complementary order property. If q_1, q_2 are real and $q_2 - q_1 \geq 0$, then every continuous lift of the phase decreases along the segment joining q_1 to q_2, with the exact difference

$$\Theta(1) - \Theta(0) = -\frac{k}{2} \int_0^1 \int_0^R (q_2 - q_1)(x) |u(x, k; q_1 + s(q_2 - q_1))|^2 \, dx \, ds. \tag{13.3}$$

This comparison statement is global along the coefficient segment; it does not rely on a smallness assumption.

13.2. **Scope of the analysis.** Compact support is used in two essential ways. It supplies an exact two-exponential representation beyond R, and it reduces the scattering coefficient to a quotient of entire endpoint data. Short-range, noncompactly supported coefficients should admit related threshold expansions, but their treatment requires estimates for solutions normalized at infinity and lies outside the present argument. The Taylor series proved here is local: by Theorem 11.1, its radius is limited by the nearest complex zero of the reduced boundary denominator.

The Volterra analysis remains valid for complex-valued $q \in L^1(0, R)$. Reality, unitarity, and phase monotonicity do not. The positivity condition $1 + q \geq c_0$ is needed only for the self-adjoint weighted realization in Section 2; it is not used in the coefficient calculus. For sign-changing coefficients, the displayed moments may vanish, and the first nonzero coefficient-dependent term can occur beyond the structural order identified in Remark 7.2.

The results concern a scalar equation. The convergent Taylor and Born series are proved for fixed coefficient and endpoint data, while Section 10.1 permits the prescribed frequency-dependent families q_k and $\nu(k)$. Its power-law classification is deliberately restricted to $\gamma < 2$: at $\gamma = 2$ the limiting transfer equation may be resonant, and above that power sign-changing profiles need not possess a universal limit. Higher-order uniform normal forms for singular families, matrix-valued systems, and general nonlinear spectral dependence in the boundary condition are natural further problems.

13.3. **Concluding statement.** The passage from the regular solution to the scattering coefficient is finite-dimensional at the endpoint. The Volterra equation gives convergent series for a single regular solution, and the boundary quotient converts those series into an explicit recursion for every Taylor and Born coefficient. Removing the common factor at $\nu = 0$ separates the Neumann threshold from fixed nonzero Robin data and explains both the discontinuity of the zeroth-order limit and the change from linear to cubic coefficient sensitivity. The exact two-potential identity upgrades the linearization to a finite perturbation formula, and the optimal L^1 modulus quantifies the uniform fixed-Robin/Neumann crossover. For singular contrast, the boundary parameter is renormalized by the mean of the leading profile; if that mean vanishes, the negative quadratic form $\mathcal{J}$ becomes the leading invariant and shifts the critical exponent from 1 to 3/2. These threshold laws are formulated entirely through the Sturm–Liouville and Volterra structure. In one-dimensional reductions of elliptic equations, they provide computable low-frequency boundary invariants without requiring additional smoothness of the coefficient.

Appendix A. Simplex formulas for the regular coefficients

The recursive coefficients in Section 6 admit explicit ordered-integral representations. These formulas are not needed for analyticity, but they make the polynomial dependence on q transparent. Put $w = 1 + q$ and, for $m \geq 1$, define the simplex

$$\Sigma_m(x) := \{(t_1, \dots, t_m) : 0 < t_m < \cdots < t_1 < x\}.$$

Proposition A.1 (Ordered-integral representation). *For $m \geq 1$,*

$$\begin{aligned} p_m(x;\nu,q) = (-1)^m \int_{\Sigma_m(x)} &(x - t_1)(t_1 - t_2) \cdots (t_{m-1} - t_m) \\ &\times \prod_{j=1}^{m} w(t_j)(1 + \nu t_m)\, \mathrm{d}t_m \cdots \mathrm{d}t_1. \end{aligned} \tag{A.1}$$

Moreover,

$$\begin{aligned} b_m(\nu;q) = (-1)^m \int_{\Sigma_m(R)} &(t_1 - t_2) \cdots (t_{m-1} - t_m) \\ &\times \prod_{j=1}^{m} w(t_j)(1 + \nu t_m)\, \mathrm{d}t_m \cdots \mathrm{d}t_1, \end{aligned} \tag{A.2}$$

where the empty product of difference factors for $m = 1$ equals one.

Proof. By construction,

$$p_m = (-1)^m \mathcal{V}_q^m (1 + \nu x).$$

Repeated substitution of (4.2) gives (A.1). Differentiation with respect to x removes the first factor $x - t_1$; evaluating at $x = R$ proves (A.2). The moving-boundary term vanishes because the removed factor in (A.1) is zero when $t_1 = x$. □

For $q = \varepsilon V$, expand

$$\prod_{j=1}^{m}(1+\varepsilon V(t_j)) = \sum_{\ell=0}^{m} \varepsilon^{\ell} \sum_{\substack{I\subset\{1,\dots,m\}\\ |I|=\ell}} \prod_{j\in I} V(t_j).$$

It follows that $p_m(x;\nu,\varepsilon V)$ is a polynomial in ε of degree at most m, and its coefficient of ε^{ℓ} is obtained from (A.1) by inserting the inner subset sum. Hence the coefficient of k^{2m} in the regular solution involves at most m interactions with V. This triangular relation is the algebraic compatibility between the spectral recursion of Theorem 6.2 and the Born recursion of Theorem 9.2.

Corollary A.2 (Endpoint coefficients as continuous polynomials)**.** *For every $m \geq 0$, the maps*

$$q \longmapsto a_m(\nu;q), \qquad q \longmapsto b_m(\nu;q)$$

are continuous polynomials of degree at most m on $L^1(0,R)$. Their homogeneous components are continuous multilinear integral forms on ordered simplices.

Proof. Expand the product $\prod(1+q(t_j))$ in (A.1)–(A.2). Each term is bounded by a constant, depending on m, R, ν, times $\|q\|_1^{\ell}$ for some $0 \leq \ell \leq m$. □

Appendix B. A weighted-Sobolev a posteriori estimate

The original motivation for the project included control of an approximate solution on the half-line. We give here a corrected functional-analytic statement. Unlike the preceding L^1 theory, this appendix assumes

$$q \in L^{\infty}(0,R), \qquad q = 0 \quad \text{on } (R,\infty), \tag{B.1}$$

so that multiplication preserves the weighted spaces below.

For $\beta > 0$, let

$$L^2_{\beta} := \{f : e^{\beta x} f \in L^2(0,\infty)\}, \tag{B.2}$$

$$H^2_{\beta} := \{v : e^{\beta x} v^{(j)} \in L^2(0,\infty),\ 0 \leq j \leq 2\}, \tag{B.3}$$

with their natural graph norms. Fix $\nu \neq 0$ and set

$$X_{\beta} := \mathbb{C} \times H^2_{\beta}, \qquad Y_{\beta} := \mathbb{C} \times L^2_{\beta}.$$

For $k \in \mathbb{C}$, define

$$\begin{aligned}\mathbb{L}_{k,q}(c,v) := \big(&-\mathrm{i}kc + v'(0) - \nu(c+v(0)),\\ &v'' + k^2(1+q)v + k^2 q\, c\, e^{-\mathrm{i}kx}\big).\end{aligned} \tag{B.4}$$

Lemma B.1 (The zero-frequency isomorphism)**.** *For $\nu \neq 0$, the map*

$$\mathbb{L}_{0,q}(c,v) = \big(v'(0) - \nu(c+v(0)), v''\big)$$

is an isomorphism from X_{β} onto Y_{β}. Its inverse does not depend on q.

Proof. Given $(g,f) \in Y_\beta$, set

$$v(x) := \int_x^\infty (t-x) f(t)\,\mathrm{d}t, \qquad c := \nu^{-1}\big(v'(0) - g\big) - v(0). \tag{B.5}$$

Then $v'' = f$, v and v' vanish at infinity in the weighted sense, and the first component equals g. Writing

$$v(x) = \int_0^\infty s f(x+s)\,\mathrm{d}s, \qquad v'(x) = -\int_0^\infty f(x+s)\,\mathrm{d}s,$$

Young's inequality with the kernels $se^{-\beta s}$ and $e^{-\beta s}$ gives

$$\|v\|_{H^2_\beta} \le C_\beta \|f\|_{L^2_\beta}. \tag{B.6}$$

Cauchy–Schwarz similarly bounds $v(0)$ and $v'(0)$. Thus

$$|c| + \|v\|_{H^2_\beta} \le C_{\beta,\nu}\big(|g| + \|f\|_{L^2_\beta}\big).$$

The construction proves surjectivity and the estimate; applying it to $(g,f) = (0,0)$ proves injectivity. $\square$

Theorem B.2 (Small-frequency invertibility)**.** *Under* (B.1)*, for every $\beta > 0$ and $\nu \neq 0$ there are $\kappa > 0$ and $C > 0$ such that*

$$\mathbb{L}_{k,q} : X_\beta \longrightarrow Y_\beta$$

is invertible whenever $|k| < \kappa$, and

$$|c| + \|v\|_{H^2_\beta} \le C\|\mathbb{L}_{k,q}(c,v)\|_{Y_\beta}. \tag{B.7}$$

Proof. The trace map $H^2_\beta \to \mathbb{C}^2$ is continuous. Compact support of q, together with (B.1), gives

$$\|(\mathbb{L}_{k,q} - \mathbb{L}_{0,q})(c,v)\|_{Y_\beta} \le C_q(|k| + |k|^2)\big(|c| + \|v\|_{H^2_\beta}\big)$$

for $|k| \le 1$. Lemma B.1 and the Neumann series therefore prove invertibility for sufficiently small $|k|$, as well as (B.7). $\square$

Corollary B.3 (Residual control of the coefficient)**.** *Let q, β, ν satisfy the hypotheses of Theorem B.2, let $k \in \mathbb{R}$ be sufficiently small and nonzero, and let u be the exact solution normalized by* (1.4)*. Suppose $\widetilde{u} \in H^2_{\mathrm{loc}}(0,\infty)$ and $\widetilde{S} \in \mathbb{C}$ satisfy*

$$\widetilde{u} - e^{\mathrm{i}kx} - \widetilde{S}e^{-\mathrm{i}kx} \in H^2_\beta. \tag{B.8}$$

Define the differential and boundary residuals

$$\widetilde{f} := \widetilde{u}'' + k^2(1+q)\widetilde{u} \in L^2_\beta, \tag{B.9}$$

$$\widetilde{g} := \widetilde{u}'(0) - \nu\widetilde{u}(0). \tag{B.10}$$

Then

$$|S_\nu(k;q) - \widetilde{S}| \le C\left(|\widetilde{g}| + \|\widetilde{f}\|_{L^2_\beta}\right). \tag{B.11}$$

Proof. Write

$$u-\widetilde{u}=ce^{-\mathrm{i}kx}+v,\qquad c:=S_\nu(k;q)-\widetilde{S}.$$

The exact exterior identity for u and (B.8) show that $v\in H^2_\beta$. Subtracting the exact equation and boundary condition from the approximate ones gives

$$\mathbb{L}_{k,q}(c,v)=(-\widetilde{g},-\widetilde{f}).$$

Estimate (B.11) follows from (B.7). □

Remark B.4. At $\nu=0$, the map in Lemma B.1 has the one-dimensional kernel generated by $(1,0)$. This is the weighted-space counterpart of the common factor removed in (5.6). A uniform Neumann estimate must first separate that threshold mode; the fixed-Robin estimate above cannot be continued to $\nu=0$ with a bounded constant.

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REFERENCES

[1] T. Aktosun, M. Klaus, and R. Weder, *Small-energy analysis for the self-adjoint matrix Schrödinger operator on the half line*, J. Math. Phys. **52** (2011), 102101, doi:10.1063/1.3640029.

[2] T. Aktosun, M. Klaus, and R. Weder, *Small-energy analysis for the selfadjoint matrix Schrödinger operator on the half line. II*, J. Math. Phys. **55** (2014), 032103, doi:10.1063/1.4866780.

[3] T. Aktosun and C. van der Mee, *Scattering and inverse scattering for the 1-D Schrödinger equation with energy-dependent potentials*, J. Math. Phys. **32** (1991), 2786–2801, doi:10.1063/1.529070.

[4] D. Bollé, F. Gesztesy, and S. F. J. Wilk, *A complete treatment of low-energy scattering in one dimension*, J. Operator Theory **13** (1985), 3–32.

[5] P. Deift and E. Trubowitz, *Inverse scattering on the line*, Comm. Pure Appl. Math. **32** (1979), 121–251, doi:10.1002/cpa.3160320202.

[6] S. Demirel and M. Usman, *Trace formulas for Schrödinger operators on the half-line*, Bull. Math. Sci. **1** (2011), 397–427, doi:10.1007/s13373-011-0016-1.

[7] R. O. Hryniv and S. S. Manko, *Inverse scattering on the half-line for energy-dependent Schrödinger equations*, Inverse Problems **36** (2020), 095002, doi:10.1088/1361-6420/aba416.

[8] Y. Kamimura, *Energy dependent inverse scattering*, Funkcial. Ekvac. **51** (2008), 189–219, doi:10.1619/fesi.51.189.

[9] T. Kato, *Perturbation Theory for Linear Operators*, Classics in Mathematics, Springer, Berlin, 1995.

[10] E. Korotyaev, A. Mantile, and D. Mokeev, *Inverse resonance problems for energy-dependent potentials on the half-line*, SIAM J. Math. Anal. **56** (2024), 2115–2148, doi:10.1137/23M1591104.

[11] V. V. Kravchenko and R. M. Porter, *Spectral parameter power series for Sturm–Liouville problems*, Math. Methods Appl. Sci. **33** (2010), 459–468, doi:10.1002/mma.1205.

[12] V. V. Kravchenko, S. M. Torba, and U. Velasco-García, *Spectral parameter power series for Sturm–Liouville equations with a potential polynomially dependent on the spectral parameter and Zakharov–Shabat systems*, J. Math. Phys. **56** (2015), 073508, doi:10.1063/1.4927253.
[13] F. Loran and A. Mostafazadeh, *Low-frequency scattering defined by the Helmholtz equation in one dimension*, J. Phys. A: Math. Theor. **54** (2021), no. 31, 315204, doi:10.1088/1751-8121/ac019e.
[14] B. Simon, *The bound state of weakly coupled Schrödinger operators in one and two dimensions*, Ann. Physics **97** (1976), no. 2, 279–288, doi:10.1016/0003-4916(76)90038-5.
[15] G. Teschl, *Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators*, Graduate Studies in Mathematics, vol. 99, American Mathematical Society, Providence, RI, 2009.
[16] J. Weidmann, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Mathematics, vol. 1258, Springer, Berlin, 1987.
[17] D. P. Winebrenner and J. Sylvester, *Linear and nonlinear inverse scattering*, SIAM J. Appl. Math. **59** (1998), 669–699, doi:10.1137/S0036139997319773.
[18] D. R. Yafaev, *Mathematical Scattering Theory: General Theory*, Translations of Mathematical Monographs, vol. 105, American Mathematical Society, Providence, RI, 1992.

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