Question 1

(a) (**3 marks**) Let G be a graph and $k \geq 1$ an integer. What does it mean to say that (i) G is *k-colourable*; (ii) G has *chromatic number* $\chi(G) = k$.

(b) (**4 marks**) Give an example of a triangle-free graph G with $\chi(G) = 3$.

(c) (**7 marks**) Prove that if G has maximum degree $\Delta(G)$, then $\chi(G) \leq \Delta(G) + 1$.

(d) (**11 marks**) Prove that if G is a graph with m edges, then G contains a 3-colourable subgraph with at least $\lceil 2m/3 \rceil$ edges.

Question 2

(a) (**2 marks**) Define $\mathrm{ex}(n, H)$, where H is a graph and $n \geqslant 1$ is an integer.

(b) (**5 marks**) Define the Turán graph $T_r(n)$. How many edges are there in $T_4(11)$?

(c) (**8 marks**) Given a graph H define $\pi(H)$, the Turán density of H, and prove that $\pi(H)$ is well-defined.

(d) (**10 marks**) Show that if G is a K_3-free graph of order $2n$ with $n^2 - t$ edges, for some $t \geqslant 0$, then G contains a bipartite subgraph with at least $n^2 - 2t$ edges.

Question 3

(a) (**3 marks**) What does it mean to say that a family of sets is (i) an *antichain*; (ii) a *chain*.

(b) (**5 marks**) Prove that if $n \geqslant 1$ is an integer and C is an chain in $\mathcal{P}([n])$, then

$$|C| \leqslant n + 1.$$

(c) (**7 marks**) Give a symmetric chain decomposition of $\mathcal{P}([4])$.

(d) (**3 marks**) State Sperner's Theorem.

(e) (**7 marks**) Let $X = \{x_1, x_2, x_3, x_4, x_5\} \subseteq [1, \infty)$. Let us call $\alpha \in \mathbb{R}$ an X-sum if there exists $A \subseteq X$ such that $\sum_{a \in A} a = \alpha$. Prove that if $\alpha_1, \ldots, \alpha_{11}$ are 11 distinct X-sums, then there exists i, j such that $\left|a_i - a_j\right| \geqslant 1$.

Question 4

(a) (**10 marks**) Let $s, t \geqslant 2$ be integers. Define the Ramsey number $R(s, t)$ and prove that it satisfies

$$R(s, t) \leqslant \binom{s + t - 2}{s - 1}.$$

(b) (**5 marks**) Let $R_k(3)$ be the smallest integer n such that any colouring of the edges of K_n with k colours contains a monochromatic K_3. Using the fact that $R(3, 3) = 6$ or otherwise show that $R_3(3) \leqslant 17$.

(c) (**10 marks**) Prove that if $n \geqslant s$ satisfy

$$\binom{n}{s} < 4^{\binom{s}{2} - 1},$$

then $R_4(s) > n$.